2020 VCAA Mathematical Methods Exam 2 Solutions

Use CAS whenever practical

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
D	E	C	D	E	B	C	B	E	B

11	12	13	14	15	16	17	18	19	20
C	E	A	A	B	D	C	D	A	B

Q1 $g(f(-1)) = g(4) = 6$ D

Q2 $p(-2) = (-2)^3 - 2a(-2)^2 + (-2) - 1 = 5$, $a = -2$ E

Q3 C

Q4 $2x - \frac{\pi}{3} = 2k\pi - \frac{2\pi}{3}, 2k\pi + \frac{2\pi}{3}$, $x = \frac{\pi(6k-1)}{6}, \frac{\pi(6k+3)}{6}$ D

Q5 E

Q6 $f'(x) = 0$ at two values of $x < 0$ B

Q7 C

Q8 $n = 25$, $p = \frac{1.4}{25} = 0.056$, $\Pr(X > 3) \approx 0.048$ B

Q9 $\int_4^8 f(x)dx = 2\int_2^4 f(2x)dx = 2\int_0^2 f(2(x+2))dx = 5$ E

Q10 $n + 1 = 2^x$, $n = 2^x - 1 = 2^k - 1$ for $k \in Z^+$ B

Q11 $\Pr(X < 259) = \Pr\left(Z < \frac{259-250}{\sigma}\right) = \Pr(Z < 1.5)$, $\sigma = 6$ C

Q12 Amplitude $= 10$, $h(0) = 25$, period $= 60 = \frac{2\pi}{n}$.: $n = \frac{\pi}{30}$ E

Q13 Left translation by 4, then horizontal dilation by factor $\frac{1}{2}$ A

Q14 $\Pr(X > 5.2) = \Pr\left(Z > \frac{5.2 - 2\sigma}{\sigma}\right) = \Pr\left(\frac{5.2}{\sigma} - 2\right) = 0.9$,

$\sigma \approx 7.238$ A

Q15 Average value $= \frac{\frac{1}{2}(3a)(2a) + \frac{1}{2}(2a)(a) - a(3a)}{3a} = \frac{a}{3}$ B

Q16 $A = \frac{1}{2}m(9 - m^2)$, let $\frac{dA}{dm} = 0$, $C(\sqrt{3}, 6)$, max area $= 3\sqrt{3}$ D

Q17 $f(0) = -\log_e 2$, a tangent at $x = 0$ has $c = -\log_e 2$.
Tangents at $x \neq 0$ have $c < -\log_e 2$ C

Q18 D

Q19 $p(w) = {}^{20}C_w \left(\frac{1}{6}\right)^w \left(\frac{5}{6}\right)^{20-w}$

$q(w) = {}^{20}C_w \left(\frac{5}{6}\right)^w \left(\frac{1}{6}\right)^{20-w} = p(20 - w)$ A

Q20 $\cos ax = \cos a(x + h)$, $T = 1$.: $a = 2\pi$,

$-1 \le \log_2(\cos 2\pi x) \le 0$.: $\frac{1}{2} \le \cos 2\pi x \le 1$, $\frac{5}{6} \le x \le \frac{7}{6}$ B

SECTION B

Q1a $f(0) = 16a = 4$, $a = \frac{1}{4}$

Q1b $f(x) = \frac{1}{4}(x^2 - 4)^2 = \frac{1}{4}(x^4 - 8x^2 + 16) = \frac{1}{4}x^4 - 2x^2 + 4$

Q1ci $f'(x) = x^3 - 4x$

Q1cii $f''(x) = 0$ and $x \in (0, 2)$, $3x^2 - 4 = 0$, $x = \frac{2}{\sqrt{3}}$

.: minimum $f' = f'\left(\frac{2}{\sqrt{3}}\right) = -\frac{16}{3\sqrt{3}}$

Q1d Reflection in the x-axis
Translation of 2 units upwards

Q1ei The two graphs intersect at $y = 1$.

.: $\frac{1}{4}(x+2)^2(x-2)^2 = 1$, $x = \pm\sqrt{2}, \pm\sqrt{6}$

Q1eii $\int_{\sqrt{2}}^{\sqrt{6}} 4\left(1 - \frac{1}{4}(x+2)^2(x-2)^2\right)dx$

Q1eiii Total area ≈ 2.72

Q1f $D \le 2$ inside the shaded regions.
Outside the shaded regions, let $\frac{1}{4}(x+2)^2(x-2)^2 - 1 = 1$

.: $(x+2)^2(x-2)^2 = 8$, $x \approx -2.61, -1.08, 1.08, 2.61$

.: $-2.61 \le x \le -1.08$ or $1.08 \le x \le 2.61$, shaded regions included.

Q2a f_1 is the vertical translation of f_2 by 10 m.
Swim distance $= 10$ m

Q2b $f_1(x) = 20\cos\frac{\pi x}{100} + 40 = 30$, $\cos\frac{\pi x}{100} = -\frac{1}{2}$, $\frac{\pi x}{100} = \frac{2\pi}{3}$

.: $x = \frac{200}{3}$. Swim distance $= \frac{200}{3} - 50 = \frac{50}{3}$ m

Q2c Distance $D = \sqrt{(x-50)^2 + \left(20\cos\frac{\pi x}{100} + 40 - 30\right)^2}$

$D_{\min} \approx 8.5$ m

Q2d Area $= 10 \times 200 = 2000\ \text{m}^2$

Q2e The horizontal line cuts f_1 at $x = \frac{200}{3}$ and $x = \frac{400}{3}$

Area $= \int_{50}^{150} 30 - f_2(x)dx - \int_{\frac{200}{3}}^{\frac{400}{3}} 30 - f_1(x)dx \approx 837\ \text{m}^2$

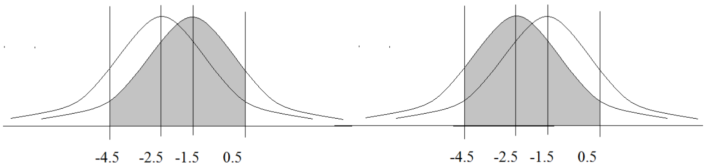
Q2f Given $k \ge 1$ and $kf_1(0) - f_2(0) < 20$, $k(20 + 40) - (20 + 30) < 20$

.: $1 \le k < \frac{7}{6}$

Q3a $a \approx 1$

Q3b $\Pr(T \le 3 \mid T > 0) = \dfrac{\Pr(0 < T \le 3)}{\Pr(T > 0)} \approx 0.547$

Q3c $k = -2.5, -1.5$



Q3d $n = 8, \quad p = 0.85 \quad \Pr(N \le 3) \approx 0.003$

Q3ei $p = 0.15$, $\Pr(N \ge 1) = 1 - \Pr(N = 0) = 1 - 0.85^n$

Q3eii $1 - 0.85^n \ge 0.95$, $n = 19$

Q3f $xy + 0.85(1 - y) = 0.75$, use $x = 0.3, 0.7$ to find the minimum and maximum values of y respectively.

$y_{\min} = \dfrac{2}{11}$, $y_{\max} = \dfrac{2}{3}$

Q4a $f(x) = 2xe^{1-x^2}, 0 \le x \le 3$

$f'(x) = 2e^{1-x^2} + 2x(-2x)e^{1-x^2}$, $f'(1) = -2$

Q4b $\tan\theta = -2$, $\theta \approx 117°$

Q4c $f'(p) = 2e^{1-p^2}\left(1 - 2p^2\right)$

Q4di $f'(1)f'(p) = (-2)\left(2e^{1-p^2}\right)\left(1 - 2p^2\right) = -1$, $p \approx 0.655$

Q4dii Tangent at $x = 1$, $y = f(1) = 2$, $y = -2(x-1) + 2$

Tangent at $x = p$, $y = 2pe^{1-p^2}$,

$y = 2e^{1-p^2}\left(1 - 2p^2\right)(x - p) + 2pe^{1-p^2}$

Perpendicular when $p \approx 0.655$

Solve simultaneously, $x \approx 0.80$, $y \approx 2.39$

Q4ei $y = \dfrac{2ne^{1-n^2}}{n}x$, $y = 2e^{1-n^2}x$. Let $p(x) = 2e^{1-n^2}x$

Q4eii $y = \dfrac{f(3) - f(n)}{3 - n}(x - 3) + f(3)$

.: $y = \dfrac{6e^{-8} - 2ne^{1-n^2}}{3 - n}(x - 3) + 6e^{-8}$

Let $q(x) = \dfrac{6e^{-8} - 2ne^{1-n^2}}{3 - n}(x - 3) + 6e^{-8}$

Q4eiii Solve $\int_0^n (f(x) - p(x))dx = \int_n^3 (q(x) - f(x))dx$ for n,

$n \approx 1.088$

Q5a $f(x) = x^3 - x$, $f'(x) = 3x^2 - 1$

Tangent to the graph of $f(x)$ at $x = a$: $y - f(a) = f'(a)\,(x - a)$

x-intercept $(b, 0)$.: $0 - \left(a^3 - a\right) = \left(3a^2 - 1\right)(b - a)$.: $b = \dfrac{2a^3}{3a^2 - 1}$

Q5b $3a^2 = 1$, $a = \pm\dfrac{1}{\sqrt{3}}$

Q5c The graph of g_a is a horizontal line when b does not exist.

Q5di $b = \dfrac{2a^3}{3a^2 - 1} = 1.1$, $a = -0.5052, 0.8084, 1.3468$

Q5dii $(-0.505, -0.5] \cup (0.808, 1.347)$

Q5e The two lines are parallel when $f'(a) = f'(b)$, $a^2 = b^2$

Given $b \ne a$.: $b = -a$

Let $\dfrac{2a^3}{3a^2 - 1} = -a$.: $a = \pm\dfrac{1}{\sqrt{5}}$

Q5f $p(-x) = (-x)^3 + w(-x) = -p(x)$

Q5g For $w \ge 0$, $p(x) = x^3 + wx$ has only one x-intercept at $x = 0$ and curves upwards. Tangent at $x > 0$ cuts the x-axis at $x > 0$.

For $w < 0$, $p(x)$ has x-intercepts at $x = 0$, $x = \pm\sqrt{-w}$, the same general features as $f(x)$ and satisfies the parallel tangents property as in part e.

Q5h m and n are dilations/reflections of function p. They do not change the parallel tangents property.

h and k are translations. If the tangent at $x = t$ is not required to cut the x-axis at $x = -t$, then $h = 0$ and $k \in R$, otherwise $h = k = 0$.

Please inform admin@itute.com re conceptual and/or mathematical errors

