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2021 Specialist Mathematics

Year 12 Application Task

(Time allowed: 4 hours plus)

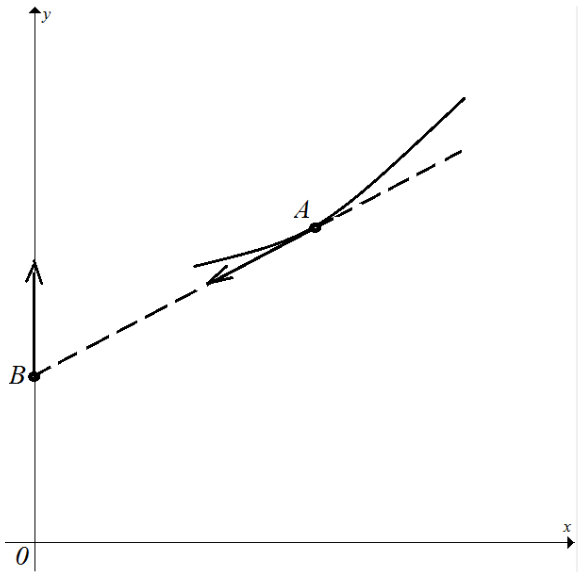
Theme: Jet-ski chase

The rider of a jet-ski sees another jet-ski travelling in a straight line and gives chase.
The task is to explore the path of a jet-ski in pursuit of another.

Assumed knowledge: Position vectors, parametric equations, differentiation, integration, differential equations, length of an arc, kinematics, graphs, CAS

Part I (80-90 min)

Jet-ski A has position vector $\tilde{r}_A(t) = x_A(t)\tilde{i} + y_A(t)\tilde{j}$ at time t.
Jet-ski B has position vector $\tilde{r}_B(t) = y_B(t)\tilde{j}$ at time t.
Jet-ski A gives chase to Jet-ski B.
Jet-ski A always heads straight for Jet-ski B.
Jet-ski A travels at the same speed as Jet-ski B.



Let $\tilde{v} = \dfrac{d\tilde{r}}{dt} = \dot{r} = \dot{x}\tilde{i} + \dot{y}\tilde{j}$.

a. Write an expression for $|\tilde{v}_A|$ in terms of the $\tilde{i}$ and $\tilde{j}$ components of $\tilde{v}_A$.

b. Write an expression for $|\tilde{v}_B|$ in terms of the component of $\tilde{v}_B$.

c. Write an expression for $|\tilde{r}_B - \tilde{r}_A|$ in terms of the $\tilde{i}$ and $\tilde{j}$ components of $\tilde{r}_B - \tilde{r}_A$.

d. $\underset{\sim}{v}_A$ makes angle θ with the x-axis. Find θ in terms of the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of $\underset{\sim}{v}_A$.

e. $\underset{\sim}{r}_B - \underset{\sim}{r}_A$ makes the same angle θ with the x-axis.
Find θ in terms of the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of $\underset{\sim}{r}_B - \underset{\sim}{r}_A$.

f. Show that $\dfrac{\dot{y}}{\dot{x}} = \dfrac{dy}{dx}$.

g. Using some/all results in parts a to f, show that $y_B = y_A - x_A \dfrac{d\,y_A}{d\,x_A}$.

h. Hence show that $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = x\dfrac{d^2y}{dx^2}$ for Jet-ski A, i.e. $\sqrt{1+\left(\dfrac{d\,y_A}{d\,x_A}\right)^2} = x_A \dfrac{d^2 y_A}{d\,x_A^2}$

i. Show that $\frac{d}{du}\log_e\left(u+\sqrt{1+u^2}\right)=\frac{1}{\sqrt{1+u^2}}$.

Let $u=\frac{dy}{dx}$.

$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=x\frac{d^2y}{dx^2}$ can be written as $\sqrt{1+u^2}=x\frac{du}{dx}$.

At point $(1,\beta)$, $u=\frac{dy}{dx}=0$ for Jet-ski A.

j. Without using CAS show that for $\beta=0$, $\sqrt{1+u^2}=x\frac{du}{dx}$ has solution given by $u+\sqrt{1+u^2}=x$.

k. Hence, without using CAS, show that for $\beta=0$, $y=\frac{x^2-1}{4}-\frac{1}{2}\log_e x$.

l. Sketch the path of Jet-ski A, $y=\frac{x^2-1}{4}-\frac{1}{2}\log_e x$ for $\beta=0$.

m. Sketch the paths of Jet-ski A for two other real values of $\beta \in [1, 3]$ on the same set of axes in the previous part. Label each path with its value of β.

n. Comment on the effects of changing the value of β on the path of Jet-ski A.

Part II (80-90 min)

If students attempt Part II at a latter time, they should have copies of their Part I to access the given information and their workings.

Continuation of Part I:

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = x\frac{d^2y}{dx^2}$ can be written as $\sqrt{1+u^2} = x\frac{du}{dx}$ where $u = \frac{dy}{dx}$.

Now, for Jet-ski A, $u = \frac{dy}{dx} = 0$ at point $(\alpha, 0)$.

a. Without using CAS show that for $\alpha = 2$, $\sqrt{1+u^2} = x\frac{du}{dx}$ has solution given by $u + \sqrt{1+u^2} = \frac{x}{2}$.

b. Hence, without using CAS, show that for $\alpha = 2$, $y = \frac{x^2-4}{8} - \log_e \frac{x}{2}$.

c. Show that for $\alpha \in R$, $y = \frac{x^2-\alpha^2}{4\alpha} - \frac{\alpha}{2}\log_e \frac{x}{\alpha}$.

d. Sketch the path of Jet-ski A for three different values of $\alpha \in [1, 4]$ on the same set of axes. Label each path with its value of α.

e. Comment on the effects of changing the value of α on the path of Jet-ski A.

From part f to part j, consider the path of Jet-ski A, given $\frac{dy}{dx} = 0$ at point $(3, 0)$.

f. When Jet-ski A travels from $x = 7$ to $x = 0.5$, Jet-ski B travels from y_1 to y_2.
Find the values of y_1 and y_2.

g. Calculate the length of the path of Jet-ski A from $x = 7$ to $x = 0.5$.

h. Compare and comment on the answer in part g with the value of $y_2 - y_1$.

i. Discuss/explain whether Jet-ski A will catch up with Jet-ski B.

j. Find the shortest distance between Jet-ski A and Jet-ski B if it exists.

k. Let $y = f(x)$ be the path of Jet-ski A and $\dfrac{dy}{dx} = 0$ at point (α, β).
Find $f(x)$ in terms of parameters α and β.

l. Discuss/explain whether your answers to part i and part j are affected by changing the value of α or β.

Part III (80-90 min)

If students attempt Part III at a latter time, they should have copies of their Part I and Part II to access the given information and their workings.

In Part I and Part II
Jet-ski A has position vector $\tilde{r}_A(t) = x_A(t)\tilde{i} + y_A(t)\tilde{j}$ at time t,
Jet-ski B has position vector $\tilde{r}_B(t) = y_B(t)\tilde{j}$ at time t,
Jet-ski A always heads straight for Jet-ski B,
Jet-ski A travels at the same speed as Jet-ski B.

In Part III assume that Jet-ski A travels at a higher speed than Jet-ski B,
i.e. speed of $B = n \times$ speed of A where $0 < n < 1$.

$\therefore\ n\sqrt{1+\left(\frac{dy}{dx}\right)^2} = x\frac{d^2y}{dx^2}$ and it can be written as $n\sqrt{1+u^2} = x\frac{du}{dx}$ where $u = \frac{dy}{dx}$.

Now, for Jet-ski A, let $\frac{dy}{dx} = 0$ at point $(1, 0)$.

a. Without CAS show that $u = \frac{dy}{dx} = \frac{x^n}{2} - \frac{1}{2x^n}$.

b. Hence, without CAS, show that $y = \frac{(1-n)x^{1+n} - (1+n)x^{1-n} + 2n}{2(1-n^2)}$.

c. If the speed of Jet-ski *A* is twice the speed of Jet-ski *B*, find the point where Jet-ski *A* catches up with Jet-ski *B*. Explain/verify your answer.

d. If the speed of Jet-ski *A* is twice the speed of Jet-ski *B*, and it starts the chase at $x = 10$, find the total distance travelled during the chase.

e. Show that the distance travelled by Jet-ski *B* is a half of that travelled by Jet-ski *A* during the chase.

f. Investigate and comment on the effects of changing the value of n on (i) the path of Jet-ski *A* and (ii) the point where Jet-ski *A* catches up with Jet-ski *B*.
Suggestions: Find the equations of the path for three different values of $n \in [0.3, 0.9]$.
Sketch and label the path for each of your chosen n values on the same set of axes.

g. Let the speed of $B = \frac{1}{2} \times$ the speed of A, .: the path of Jet-ski A, $y = f(x)$ satisfies the differential equation $\frac{1}{2}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = x\frac{d^2y}{dx^2}$.

Given $\frac{dy}{dx} = 0$ at point $(\alpha, 0)$, solve the differential equation in terms of α.

h. Investigate and comment on the effects of changing the value of α on (i) the path of Jet-ski *A* and (ii) the point where Jet-ski *A* catches up with Jet-ski *B*.
Sketch and label the path for each value of α on the same set of axes.

End of Task