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2022 Specialist Mathematics

Year 12 Modelling Task

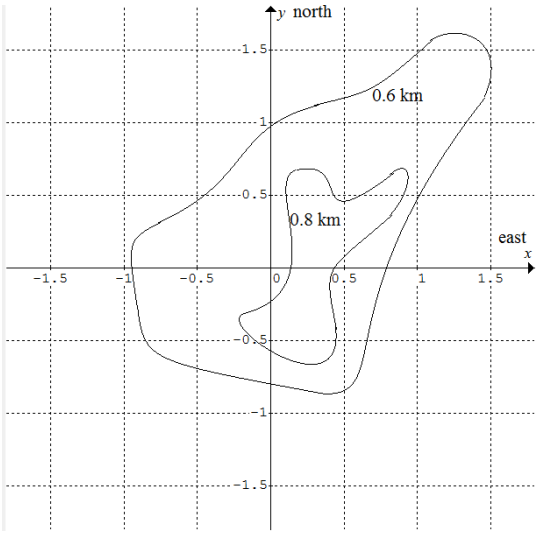
(Time allowed: 2.0 hours plus)

Modelling Task

Theme: Mountains, lakes and contour maps

Assumed knowledge:
Functions, relations, graphs, calculus, gradient, length of curve, volume of solid of revolution, and use of CAS

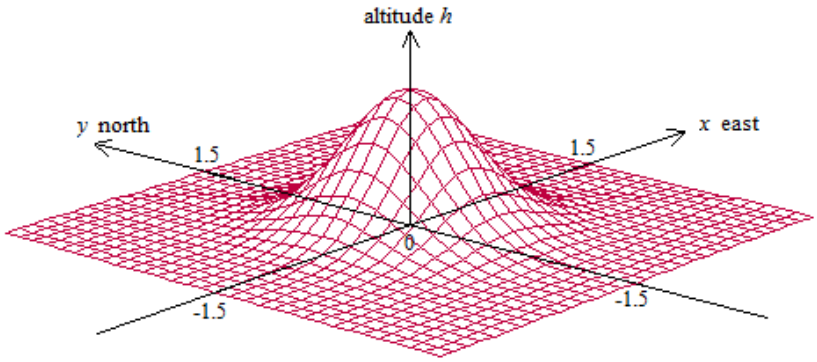
Specifications:
x and y axes are at sea level.
1 on each axis represents 1 km.
North is in the positive y direction and east is in the positive x direction.
Altitude h is height in km measured from sea level.
The following diagram is an example of a contour map showing two closed contour curves.
Points (x, y) on the same curve in a contour map are at the same altitude.



Part I (80 minutes plus)
Correct numerical answers to 4 decimal places unless stated otherwise.

The diagram below shows a 3-D picture of a mountain. Not drawn to scale.

The altitude in km at point (x, y) can be calculated using the relation $h = \frac{1}{2}e^{1-(x^2+y^2)}$.



a. Determine the altitude at the summit of the mountain.

b. Calculate the altitude in **metres** at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

c. Calculate the gradient of the slope of the mountain at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

d. In terms of h, find the gradient of the slope of the mountain where the altitude is h km.

e. Find the average gradient (magnitude only) of the slope towards the summit from $h = \frac{1}{5}$ to $h = \frac{1}{2}$.

f. Determine the steepest slope of the mountain.

g. Find $\left\{(x, y) : \frac{1}{2} e^{1-\left(x^2+y^2\right)} = \frac{1}{2}\right\}$.

h. Sketch a closed contour curve on the grid below for altitude $\frac{1}{2}$ km.

i. Find the equation of a closed contour curve on the map for altitude h km. Express $x^2 + y^2$ in terms of h. Hence find the area enclosed by the contour curve in terms of h.

A road is planned to run from west to east directly below the summit.
From $x = -3$ to $x = -1.5$ and from $x = 1.5$ to $x = 3$ the road sections follow the landscape of the regions.
From $x = -1.5$ to $x = 1.5$ the road is in a tunnel through the mountain.
The proposed model of the road section inside the tunnel is altitude $h = c - nx^4$.
The road sections are joined smoothly.

j. Show that parameters $n \approx 0.0318$ and $c \approx 0.3045$

k. Calculate the total length of the road from $x = -3$ to $x = 3$.

End of Part I

Specifications:
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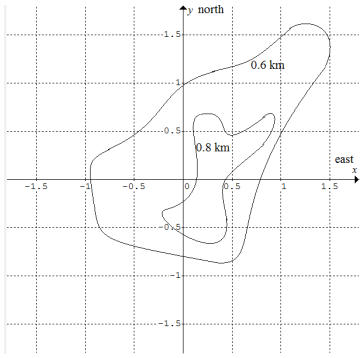
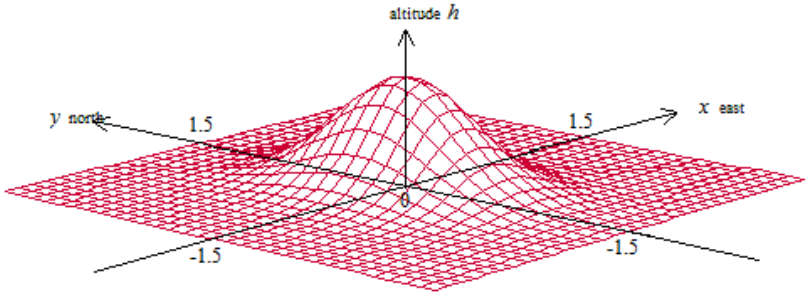
Altitude h is height in km measured from sea level.
The following diagram is an example of a contour map showing two closed contour curves.
Points (x, y) on the same curve in a contour map are at the same altitude.



Part II (80 minutes plus)
Correct numerical answers to 4 decimal places unless stated otherwise.

The diagram below shows a 3-D picture of another mountain. Not drawn to scale.

The altitude in km at point (x, y) can be calculated using the relation $h = \frac{1}{2}e^{1-(2x^2+y^2)}$.



a. Find $\left\{(x, y): \frac{1}{2}e^{1-(2x^2+y^2)} = \frac{1}{2}\right\}$.

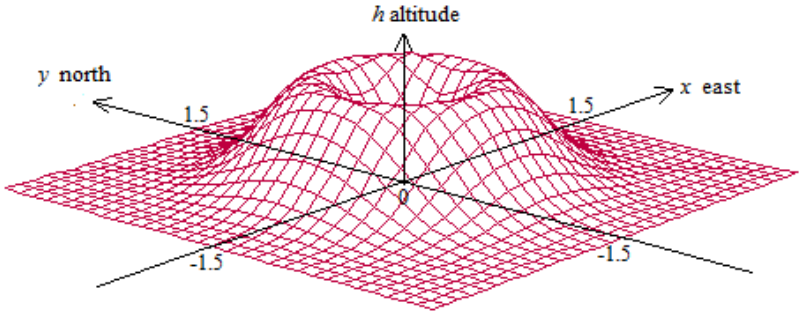
b. Sketch a closed contour curve of the mountain on the grid below for altitude $\frac{1}{2}$ km.

c. Show that the equation of a closed contour curve on the map for altitude h km is $2x^2 + y^2 = 1 - \log_e(2h)$.
Find the area enclosed by the contour curve in terms of h.
Given information: Area enclosed by an ellipse centred at $(0, 0)$ *is given by* $A = \pi ab$ *where* $a, b > 0$ *are axis intercepts.*

d. Find $\frac{dy}{dx}$ of the closed contour curve for altitude h km in part c.
Hence write a definite integral for the length of the contour curve for altitude h km.

e. Find the length of the contour curve for altitude $\frac{1}{2}$ km.

The diagram below shows a 3-D picture of a mountain with a crater. Not drawn to scale.
The altitude in km at point (x, y) is given by the relation $h = \left(x^2 + y^2 + k\right)e^{1-\left(x^2+y^2\right)}$ where $\frac{1}{20} \le k < 1$.



f. Determine the altitude of the lowest point in the crater in terms of k.

g. Determine the altitude of the highest point of the mountain in terms of k.

h. Investigate the effects of changing the value of k on the mountain and the crater.
Hint: Choose 3 suitable values of k.
Sketch the side elevation of the mountain and crater and label with its k value.
Use equal scale for vertical and horizontal axes.
The sketch should show 3 km on each side of the mountain and crater.
Comment

Consider the mountain and crater for $k = \frac{1}{5}$.

The crater is filled with water. The water surface is at altitude h. Choose a value for $h \in (0.73, 0.98)$ and use it to answer part i to part k.

i. Determine the area of the water surface.

j. Determine the volume of water in the crater in km^3, then convert it to m^3.

k. $0.02\ km^3$ of rainwater is expected to run into the crater. Determine the **rise** in water level in the crater.

End of Part II