2024 Mathematical Methods Trial Exam 1 Solutions © 2024 itute

Question 1

a. Domain $[-3, 6]$; range $[9, 27]$

b. $y = ax^2 + b$, $9 = a(-3)^2 + b$ and $27 = a(6)^2 + b$, $a = \frac{2}{3}$, $b = 3$

OR $y = 2x^2 + 1 \to \frac{y}{3} = 2\left(\frac{x}{3}\right)^2 + 1$, $y = \frac{2}{3}x^2 + 3$, $a = \frac{2}{3}$, $b = 3$

c. $y = \frac{2}{3}x^2 + 3 \to \frac{y}{\frac{1}{3}} = \frac{2}{3}\left(\frac{x}{\frac{1}{3}}\right)^2 + 3$, $y = 2x^2 + 1$

Question 2

a. An example: $\sin(2x) = -\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$

b. $\sin\left(2x + \frac{3\pi}{4}\right) = -\sin\left(2x - \frac{\pi}{4}\right)$ by symmetry .: $\sin\left(2x - \frac{\pi}{4}\right) + \sin\left(2x + \frac{3\pi}{4}\right) = 0$ for $x \in R$

c. $\sin\left(\frac{\pi}{2} - 2x\right) + \sin\left(\frac{\pi}{2} - 3x\right) = 2 \to \cos(2x) + \cos(3x) = 2$; $\cos(2x)$ and $\cos(3x)$ have periods of π and $\frac{2}{3}\pi$ respectively;
$\cos(2x)$ completes 2 cycles in 2π and $\cos(3x)$ completes 3 cycles in 2π .: their peaks coincide every 2π
.: $\sin\left(\frac{\pi}{2} - 2x\right) + \sin\left(\frac{\pi}{2} - 3x\right) = 2$ for $x = k(2\pi)$ where k is an integer.

Question 3

a. $y = \log_e(kx^n)$, inverse is $x = \log_e(ky^n)$, $ky^n = e^x$, $e^{\log_e k}y^n = e^x$, $y^n = e^{x - \log_e k}$, $y = e^{\frac{1}{n}(x - \log_e k)}$, .: $a = \frac{1}{n}$ and $b = \log_e k$

b. $y = \log_e(kx^n)$ and $y = e^{\frac{1}{n}(x - \log_e k)}$ always intersect on the line $y = x$. If they intersect at exactly one point, $y = x$ is a tangent to both curves and $\frac{dy}{dx} = 1$, $y = \log_e(kx^n) = \log_e k + n\log_e x$, $\frac{dy}{dx} = \frac{n}{x} = 1$ when $x = n$.: $n = \log_e(kn^n)$, $e^n = kn^n$, $k = \left(\frac{e}{n}\right)^n$.

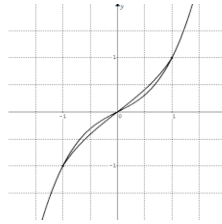
Question 4

a. $f(x) = \frac{3\sqrt{3}}{2}(x^3 - x) + 1$, $f'(x) = \frac{3\sqrt{3}}{2}(3x^2 - 1)$ Let $f'(x) = 0$.: $\frac{3\sqrt{3}}{2}(3x^2 - 1) = 0$, $x = \pm\frac{1}{\sqrt{3}}$ and $y = 0, 2$ respectively.
$f'(-1) > 0$, $f'(0) < 0$ and $f'(1) > 0$.: $\left(-\frac{1}{\sqrt{3}}, 2\right)$ and $\left(\frac{1}{\sqrt{3}}, 0\right)$ are local maximum and local minimum respectively.

b. Point of inflection $f''(x) = 0$.: $x = 0$ and $y = 1$. Line $y = mx + c$ passes through $(0,1)$ and $(-1,2)$.: $y = -x + 1$.
$y = \frac{3\sqrt{3}}{2}(x^3 - x) + 1$ and $y = -x + 1$, $\frac{3\sqrt{3}}{2}(x^3 - x) + 1 = -x + 1$, $\frac{3\sqrt{3}}{2}(x^3 - x) + x = 0$, $x\left(\frac{3\sqrt{3}}{2}x^2 - \left(\frac{3\sqrt{3}}{2} - 1\right)\right) = 0$
.: $x = 0, \pm\sqrt{1 - \frac{2\sqrt{3}}{9}}$ and $y = 1$, $1 \mp \sqrt{1 - \frac{2\sqrt{3}}{9}}$ respectively.
Intersections: $(0,1)$, $\left(\sqrt{1 - \frac{2\sqrt{3}}{9}}, 1 - \sqrt{1 - \frac{2\sqrt{3}}{9}}\right)$, $\left(-\sqrt{1 - \frac{2\sqrt{3}}{9}}, 1 + \sqrt{1 - \frac{2\sqrt{3}}{9}}\right)$.

Question 5

a.



b. $F(x)=-\frac{4}{\pi}\log_e\left(\cos\left(\frac{\pi x}{4}\right)\right)$, $F'(x)=-\frac{4}{\pi}\times\frac{1}{\cos\left(\frac{\pi x}{4}\right)}\times\left(-\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)\right)=\tan\left(\frac{\pi x}{4}\right)$

c. $A_{\text{total}}=2\times\int_0^1\left(\tan\left(\frac{\pi x}{4}\right)-\frac{1}{2}x\left(x^2+1\right)\right)dx=2\times\left[-\frac{4}{\pi}\log_e\left(\cos\left(\frac{\pi x}{4}\right)\right)-\frac{x^4}{8}-\frac{x^2}{4}\right]_0^1=2\left(-\frac{4}{\pi}\log_e\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{8}-\frac{1}{4}\right)=\frac{4}{\pi}\log_e(2)-\frac{3}{4}$

Question 6

a. $6\times6\times6=216$ b. $\Pr(A)={}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1=\frac{2}{9}$ c. $\Pr(X\le1)=\Pr(X=0)+\Pr(X=1)={}^3C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3+{}^3C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2=\frac{20}{27}$

d. $\Pr(X=1\mid X\le2)=\frac{\Pr(X=1)}{1-\Pr(X=3)}=\frac{\frac{4}{9}}{1-\frac{1}{27}}=\frac{6}{13}$ e. $\overline{X}=0\times\Pr(X=0)+1\times\Pr(X=1)+2\times\Pr(X=2)+3\times\Pr(X=3)=1$

Question 7

a. Smaller sample: $\frac{15}{75}=\frac{1}{5}$; larger sample: $\frac{75}{300}=\frac{1}{4}$

b. The larger sample will provide a better estimate.

Population standard deviation $\approx$ sample standard deviation $=\sqrt{\frac{\frac{1}{4}\left(1-\frac{1}{4}\right)}{300}}=\frac{1}{40}$; population proportion $\approx$ sample proportion

Estimated 95% confidence interval for population proportion is $\left(\frac{1}{4}-2\times\frac{1}{40},\frac{1}{4}+2\times\frac{1}{40}\right)=(0.2,\,0.3)$

c. Sample proportion has a normal distribution for large sample size ($n=300$): $E(\hat{p})\approx p\approx\frac{1}{4}$; $sd(\hat{p})\approx\frac{1}{40}$;

95% confidence interval for $\hat{p}$ is $(0.2,0.3)$.

Number of samples with $\hat{p}>0.3$ is estimated to be $\frac{1}{2}(1-95\%)\times75=2.5\%\times75\approx2$.

d. Sample proportion has a normal distribution for large sample size ($n=75$): $E(\hat{p})\approx p\approx\frac{1}{4}$; $sd(\hat{p})\approx\sqrt{\frac{\frac{1}{4}\left(1-\frac{1}{4}\right)}{75}}=\frac{1}{20}$;

68% confidence interval for $\hat{p}$ is approximately $\left(\frac{1}{4}-1\times\frac{1}{20},\frac{1}{4}+1\times\frac{1}{20}\right)=(0.2,0.3)$.

Number of samples with $\hat{p}>0.3$ is estimated to be $\frac{1}{2}(1-68\%)\times300=16\%\times300=48$.

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