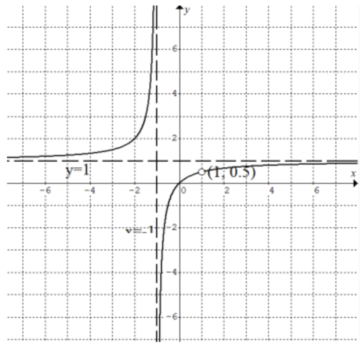
2024 Specialist Mathematics Trial Exam 1 Solutions © itute 2024

Question 1

a. $f(x)=\dfrac{x^3-2x^2+x}{x^3-x^2-x+1}=\dfrac{x(x-1)^2}{(x-1)^2(x+1)}=\dfrac{x}{x+1}=1-\dfrac{1}{x+1}$, $x\neq\pm1$ $\therefore$ $a=1$, $b=-1$ and $c=1$

b.



Question 2

a. $4x^4+5x^2+1=0$, $(4x^2+1)(x^2+1)=0$, $x^2=-\dfrac{1}{4}=\dfrac{1}{4}cis(\pi)$ or $x^2=-1=cis(\pi)$

$\therefore$ $x=\dfrac{1}{2}cis\left(\pm\dfrac{\pi}{2}\right)$ or $x=cis\left(\pm\dfrac{\pi}{2}\right)$

b. $4x^4+3x^2+1=0$, $x^2=\dfrac{-3\pm\sqrt{3^2-16}}{8}=-\dfrac{3}{8}\pm\dfrac{\sqrt{7}}{8}i=\dfrac{1}{2}cis\left(\pm\tan^{-1}\left(-\dfrac{\sqrt{7}}{3}\right)\right)$ $\therefore$ $x=\pm\dfrac{1}{\sqrt{2}}cis\left(\pm\dfrac{1}{2}\tan^{-1}\left(\dfrac{\sqrt{7}}{3}\right)\right)$

Question 3

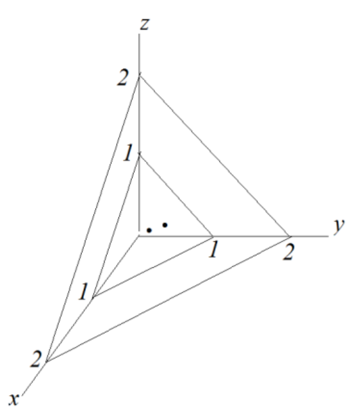
a. A: $x=0$, $y=2-t$ and $z=t-1$ $\therefore$ $x+y+z=1$ $\therefore$ A lies on the plane $x+y+z=1$

B: $x=t$, $y=2-t$ and $z=0$ $\therefore$ $x+y+z=2$ $\therefore$ B lies on the plane $x+y+z=2$

b. $\tilde{r}_B-\tilde{r}_A=t\tilde{i}-(t-1)\tilde{k}$, $|\tilde{r}_B-\tilde{r}_A|^2=t^2+(t-1)^2$. At time t, distance is $|\tilde{r}_B-\tilde{r}_A|=\sqrt{t^2+(t-1)^2}$

Closest approach, let $\dfrac{d}{dt}|\tilde{r}_B-\tilde{r}_A|^2=\dfrac{d}{dt}\left(t^2+(t-1)^2\right)=0$ $\therefore$ $t=\dfrac{1}{2}$, $|\tilde{r}_B-\tilde{r}_A|=\dfrac{1}{\sqrt{2}}$

c.



The shortest distance between the paths of the two particles is given by the shortest distance between the two planes $=$ the perpendicular distance between the two planes.

The centres of the two equilateral triangles (shown as two dots in the diagram) have position vectors $\tilde{a}=\dfrac{1}{3}\left(\tilde{i}+\tilde{j}+\tilde{k}\right)$ and $\tilde{b}=\dfrac{1}{3}\left(2\tilde{i}+2\tilde{j}+2\tilde{k}\right)$. Shortest distance between the two planes is $\left|\tilde{b}-\tilde{a}\right|=\dfrac{1}{\sqrt{3}}$.

Question 4

a. $\overrightarrow{CA}=a\tilde{i}-c\hat{k}$, $\overrightarrow{CB}=b\tilde{j}-c\tilde{k}$; $\overrightarrow{CA}\times\overrightarrow{CB}=bc\,\tilde{i}+ca\,\tilde{j}+ab\,\tilde{k}$,

area of $\Delta ABC=\dfrac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{CB}\right|=\dfrac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$

b. Unit vector perpendicular to ΔABC, $\hat{u}=\dfrac{bc\,\tilde{i}+ca\,\tilde{j}+ab\,\tilde{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$; unit vector perpendicular to ΔOAB is $\hat{v}=\tilde{k}$

Acute angle $\theta=\cos^{-1}(\hat{u}.\hat{v})=\cos^{-1}\left(\dfrac{ab}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}\right)$

Question 5

a. $(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\cdots+x^2+x^1+x^0)=x^n-1$

b. Let $x=2$, $2^n-1=(2-1)(2^{n-1}+2^{n-2}+2^{n-3}+\cdots+2^2+2^1+2^0)=2^{n-1}+2^{n-2}+2^{n-3}+\cdots+2^2+2^1+2^0$

c. Given that n is not a square number, i.e. $n\neq m^2$ where $m\in Q^+$. Suppose $\sqrt{n}=\dfrac{h}{k}$, then $h=k\sqrt{n}$ $\therefore$ $h^2=k^2n$

Since h^2 is a square number $\therefore$ k^2n must be a square number and hence $n=m^2$, a contradiction to the given information about n $\therefore$ $\sqrt{n}\neq\dfrac{h}{k}$.

Question 6

a. $\sin(x^2)+\cos(y^2)=1$, let $y=0$, $\sin(x^2)=0$, $x^2=0,\pi$; $x=0,\pm\sqrt{\pi}$, positive x-intercept $(\sqrt{\pi},0)$.

b. $\dfrac{d}{dx}\left(\sin(x^2)+\cos(y^2)\right)=\dfrac{d}{dx}(1)$, $2x\cos(x^2)-2y\sin(y^2)\dfrac{dy}{dx}=0$, let $\dfrac{dy}{dx}=\dfrac{x\cos(x^2)}{y\sin(y^2)}=0$ $\therefore$ $x\cos(x^2)=0$.

In the second quadrant, $x\neq 0$ $\therefore$ $\cos(x^2)=0$, $x^2=\dfrac{\pi}{2}$ $\therefore$ $x=-\sqrt{\dfrac{\pi}{2}}$, $y=\sqrt{\dfrac{\pi}{2}}$, stationary point $\left(-\sqrt{\dfrac{\pi}{2}},\sqrt{\dfrac{\pi}{2}}\right)$.

Question 7

a. Area $=2\times\displaystyle\int_0^2\frac{1}{\sqrt{5-y^2}}\,dy=2\left[\sin^{-1}\left(\frac{y}{\sqrt{5}}\right)\right]_0^2=2\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

b. Volume

$$=\pi\int_0^2\left(\frac{1}{\sqrt{5-y^2}}\right)^2dy=\pi\int_0^2\left(\frac{1}{\sqrt{5}+y}+\frac{1}{\sqrt{5}-y}\right)dy=\pi\left[\frac{1}{2\sqrt{5}}\left(\log_e(\sqrt{5}+y)-\log_e(\sqrt{5}-y)\right)\right]_0^2=\frac{\pi}{2\sqrt{5}}\log_e\left(\frac{\sqrt{5}+y}{\sqrt{5}-y}\right)$$

Question 8

a. $79p+84(1-p)=81$ $\therefore$ $p=0.6$

b. $\mu\approx\bar{X}=81$, $\sigma=10$ $\therefore$ $sd(\bar{X})=\dfrac{10}{\sqrt{625}}=0.4$; 82 is 2.5×0.4 from 81 $\therefore$ $\Pr(81<\bar{X}<82)\approx0.5$

$\therefore$ number of random samples out of 10 $\approx 10\times0.5=5$

Question 9

a. $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=\log_e x$

b. $\dfrac{1}{2}v^2=\int(1\times\log_e x)dx$, integration by parts by letting $1=\dfrac{dv}{dx}$ and $u=\log_e x$, $\dfrac{1}{2}v^2=x\log_e x-x+C$

Given $v=2$ when $x=e$, $\dfrac{1}{2}v^2=x\log_e x-x+2$ $\therefore$ speed $v=\sqrt{2(x\log_e x-x+2)}$.

