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Specialist Mathematics

2024

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 $-2 \le -\frac{2}{a} < 1$ if and only if

A. $a \in R \setminus [-2, 1)$

B. $a \in [-2, 1)$

C. $a \in R \setminus (-1, 2]$

D. $a \notin (-\infty, -2) \cup (1, \infty)$

E. $a \in R \setminus (-2, 1]$

Question 2 Which one of the following statements is *false*?

A. For $f(x) = \sin^2(nx)$, $\exists\, n \in R$ such that $f(x) \le 0$

B. For $f(x) = x^n$, $\exists\, n \in R$ such that $f(x) < 0$

C. For $f(x) = \sqrt[3]{nx}$, $\exists\, n \in R$ such that $f'(x) < 0$

D. For $f(x) = \cos^{-1}(nx)$, $f'(x) < 0 \;\; \forall\, n \in R^+$

E. $2^n \le n^2 - 2 \;\; \forall\, n \in R$

Question 3 The graph of $y = \sin(ax)$ from $x = \frac{2\pi}{5a}$ to $x = \frac{12\pi}{5a}$ is rotated about the x-axis.
The volume of the solid of revolution is

A. $\frac{\pi^2}{2a}$

B. $\frac{\pi^2}{a}$

C. $\frac{2\pi^2}{a}$

D. $\frac{\pi}{a^2}$

E. $\frac{2\pi}{a^2}$

Question 4 $\int_{1-\sqrt{3}}^{1} \frac{2(x-1)}{(x-3)(x+1)}\,dx =$

A. $\log_e 8 - \frac{7}{10}$

B. $\log_e 6 - \frac{2}{5}$

C. $\log_e 4$

D. $\log_e 3 + \frac{3}{10}$

E. $\frac{5}{4}\log_e 3$

Question 5 The solid bounded by the planes with equations $x = 0$, $y = 0$, $z = 0$ and $20x + 15y + 12z = 60$ has a volume of

A. 30

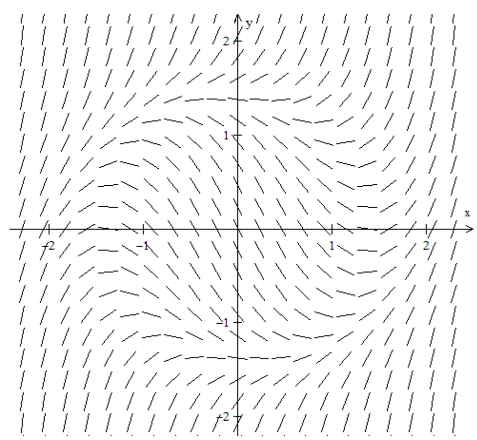
B. 20

C. 15

D. 12

E. 10

Question 6



A matching differential equation to the above direction field is

A. $\frac{dy}{dx} = x + y$

B. $\frac{dy}{dx} = \sin(x + y)$

C. $\frac{dy}{dx} = y^2 - x^2$

D. $\frac{dy}{dx} = x^2 + y^2 - 2$

E. $\frac{dy}{dx} = y - \frac{1}{e^x}$

Question 7 The equation of a plane which bisects the angle between the two planes $x=-1$ and $z=3$ is

A. $z+x=-4$

B. $y-z=2$

C. $z-x=4$

D. $x+y=2$

E. $y-z=4$

Question 8 Given $\frac{dv}{dx}=1+v^2$ for a particle in straight line motion with velocity v at position x.

The acceleration of the particle at $x=\frac{\pi}{3}$ is

A. $4\sqrt{3}$

B. $5\sqrt{2}$

C. $3\sqrt{5}$

D. $3\sqrt{7}$

E. 7

Question 9 The sum of the roots of equation $z^5+z^4+z^3+z^2+z+1=0$ is

A. $-i$

B. i

C. -1

D. 1

E. $1-i$

Question 10 $cis\left(\frac{5\pi}{14}\right)$ is a solution to $z^n=i$. Two other possible solutions are

A. $-i$, $cis\left(\frac{9\pi}{14}\right)$

B. i, $cis\left(\frac{13\pi}{14}\right)$

C. $cis\left(\frac{5\pi}{6}\right)$, $cis\left(-\frac{5\pi}{6}\right)$

D. $\frac{i}{2}$, $cis\left(\frac{5\pi}{7}\right)$

E. $cis\left(-\frac{9\pi}{14}\right)$, $cis\left(\frac{4\pi}{7}\right)$

Question 11 $y=1$ when $x=0$ for $\frac{dy}{dx}=x+y$.

By applying Euler's method with step size of 0.1, when $x=1.2$ the approximate value of y is

A. 1.51

B. 1.43

C. 1.35

D. 1.28

E. 1.20

Question 12 At time t the coordinates of a particle are $x=e^{t}-2t$ and $y=4\sqrt{2}\,e^{\frac{t}{2}}$.

The distance travelled in the interval $0 \le t \le \frac{1}{2}$ is

A. $\sqrt{e}$

B. $\log_e 5$

C. $\frac{3}{2}\log_e 3$

D. $\frac{5}{3}$

E. $\frac{3e}{5}$

Question 13 $\tilde{a}=-\tilde{i}+\tilde{j}+\sqrt{2}\,\tilde{k}$, $\tilde{b}=\tilde{i}+\sqrt{2}\,\tilde{j}-\tilde{k}$ and $c=\sqrt{2}\,\tilde{i}-\tilde{j}+\tilde{k}$ are three vectors starting from position $\tilde{r}_0=\tilde{i}+2\tilde{j}+3\tilde{k}$.

Vector $\tilde{d}$ also starts from $\tilde{r}_0$, and it makes the same *obtuse* angle with $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$.

A possible vector $\tilde{d}$ is

A. $-\frac{1}{\sqrt{3}}\left(\tilde{i}+\tilde{j}+\tilde{k}\right)$

B. $\frac{1}{\sqrt{2}}\left(\tilde{i}+\tilde{j}+\tilde{k}\right)$

C. $\frac{1}{2}\left(-\tilde{i}-\tilde{j}+\tilde{k}\right)$

D. $\frac{\sqrt{6}}{2}\left(\tilde{i}+\tilde{j}-\tilde{k}\right)$

E. $\frac{\sqrt{6}}{2}\left(\tilde{i}-\sqrt{3}\,\tilde{j}-\sqrt{2}\,\tilde{k}\right)$

Question 14 The *acute* angle made by vector $\underset{\sim}{p} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and plane $x + \frac{y}{2} + \frac{z}{3} = 1$ is closest to

A. $68°$

B. $71°$

C. $74°$

D. $77°$

E. $80°$

Question 15 Let constant position be $\underset{\sim}{r}_0 = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{r} = (\cos(t)+1)\underset{\sim}{i} + (\sin(t)+2)\underset{\sim}{j} + 3\underset{\sim}{k}$ be the position of a particle at time t. Which one of the following statements is true?

A. $\dot{\underset{\sim}{r}} \bullet \dot{\underset{\sim}{r}} = -1$

B. $\ddot{\underset{\sim}{r}} \bullet \underset{\sim}{r}_0 = \ddot{\underset{\sim}{r}} \bullet \underset{\sim}{r}$

C. $\ddot{\underset{\sim}{r}} = \underset{\sim}{r}_0 - \underset{\sim}{r}$

D. $\ddot{\underset{\sim}{r}} \bullet \underset{\sim}{r} = -1$

E. $\ddot{\underset{\sim}{r}} = \underset{\sim}{r} - \underset{\sim}{r}_0$

Question 16 A particle travels in a straight path with initial position vector $\underset{\sim}{r}_0 = 5\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and displacement vector $\underset{\sim}{s} = t\underset{\sim}{i} + 4t\underset{\sim}{j} - 2t\underset{\sim}{k}$ at time t. The path intersects plane $z = 0$ at the point with coordinates

A. $(-6, -5, 0)$

B. $(-5, -4, 0)$

C. $(-4, 5, 0)$

D. $(5, 4, 0)$

E. $(6, 5, 0)$

Question 17 The cure rate for a disease using standard medication is 65%. A new drug is created. To investigate whether the new drug is better, a null hypothesis H_0 and an alternative hypothesis H_1 are required. Which one of the following H_1 hypotheses is most appropriate?

A. The new drug does not have a cure rate over 75%

B. The new drug has a cure rate of 65%

C. The new drug does not have a cure rate of 65%

D. The new drug has a higher cure rate than 65%

E. The new drug does not have a higher cure rate than 65%

Question 18 Apples are sourced from two independent orchards A and B.

The ratio $\dfrac{\text{number of apples from orchard } A}{\text{number of apples from orchard } B} = \dfrac{a}{b}$.

Let random variable X be the weight (in grams) of an apple, $E(X) \approx 263.3$ and $Var(X) \approx 14.27$.

Also $E(X_A) = 250$, $Var(X_A) = 36$; $E(X_B) = 280$, $Var(X_B) = 16$, where X_A and X_B are the weights of apples from A and B respectively.

The value of $\dfrac{a}{b}$ is closest to

A. 1.10

B. 1.15

C. 1.20

D. 1.25

E. 1.30

Information for **Question 19** and **Question 20**

Random variable X in a large population is normally distributed. A random sample of size 625 is taken from the population. Assume that X has a mean value of 12 and a standard deviation of 5.

Question 19 $\Pr(\bar{X} > 12.1)$ is closest to

A. 0.1

B. 0.2

C. 0.3

D. 0.5

E. 0.6

Question 20 Suppose the population mean and standard deviation of X are unknown.

The sample mean and standard deviation are found to be 11.982 and 0.210 respectively.

The approximate 80% confidence interval for the mean of X is closest to

A. $(11.71, 12.25)$

B. $(11.97, 11.99)$

C. $(11.98, 12.00)$

D. $(11.98, 12.02)$

E. $(11.99, 12.05)$

SECTION B Extended-answer questions

Instructions for Section B

Answer **all** questions.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 (10 marks)

a. Prove $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\le\frac{2n-1}{n}$ by mathematical induction for $n\ge 1$. 4 marks

b. Prove the proposition: x^6-1 is divisible by 3 if and only if $x\pm 1$ is divisible by 3. 6 marks

Question 2 (14 marks)

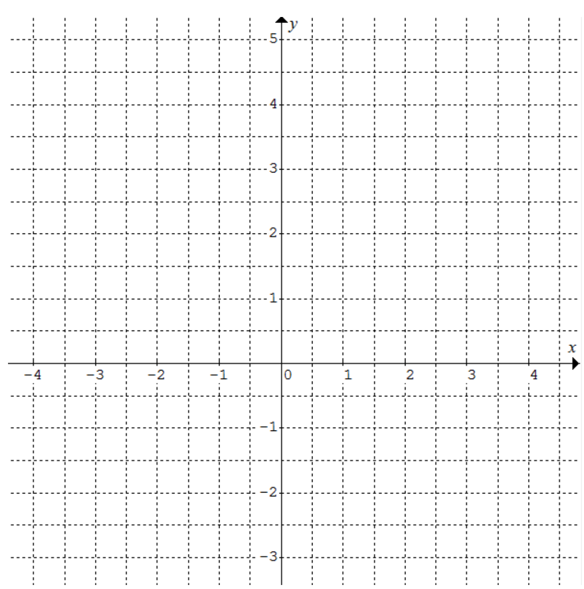
Consider rational function $f(x)=\dfrac{3}{x^3+1}$.

a. Express $f(x)$ as sum of partial fractions. 2 marks

b. Determine the exact coordinates of stationary point(s), point(s) of inflection, a and y intercepts. State the equations of asymptotes.

4 marks

c. Sketch the graph of $y=f(x)$ showing and labeling all important features. 2 marks



d. Given $\dfrac{x-2}{x^2-x+1}=\dfrac{1}{2}\left(\dfrac{ax+b}{x^2-x+1}+\dfrac{c}{x^2-x+1}\right)=\dfrac{1}{2}\left(\dfrac{ax+b}{x^2-x+1}+\dfrac{c}{(x-d)^2+e}\right)$,
find the value of each of a, b, c, d and e. 2 marks

e. Calculate the exact area between the graph of $y=f(x)$, the x-axis, $x=0$ and $x=1$. 4 marks

Question 3 (10 marks)

A particle in rectilinear motion has velocity $v=\dfrac{dx}{dt}=xt$ where position $x=1$ at time $t=0$, $0\le t\le 2$.

a. Solve the differential equation and find position x at $t=2$. 2 marks

b. Express v in terms of t only. 1 mark

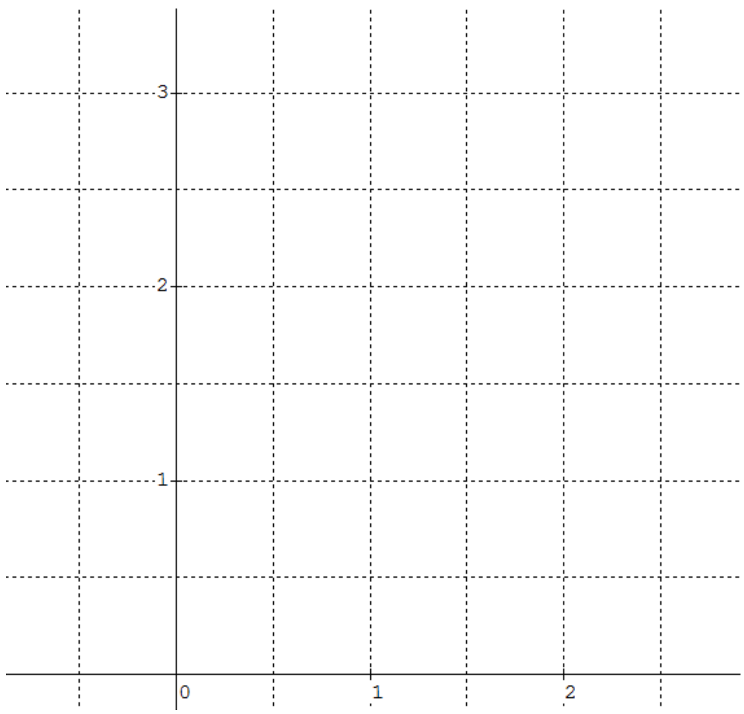
c. Express v in terms of x only. 1 mark

d. Express particle acceleration a in terms of t only. 2 marks

e. Express a in terms of x only. 1 mark

f. Sketch a direction field for $\frac{dx}{dt} = xt$ at $t = 0, 1, 2$ and $x = 1, 2, 3$. 2 marks

Use appropriate and constant length for each slope.



g. Sketch the solution curve for $\frac{dx}{dt} = xt$ on the direction field above. 1 mark

Question 4 (14 marks)

a. The rectilinear equation $a = v\frac{dv}{dx}$ can be used for curvilinear motion when v is defined as the speed (instead of velocity) of the particle moving distance x (instead of position) forwards along a curved path, and a is the rate of change in speed, i.e. $\frac{dv}{dt}$.

Let s be the distance travelled when the particle increases its speed from u to v.
Use separation of variables method, show that $v^2 = u^2 + 2as$ for constant acceleration a.

2 marks

Consider a particle in curvilinear motion.
Its position vector is $\underset{\sim}{r}(t) = 2\pi\cos\left(\frac{\pi t}{2}\right)\underset{\sim}{i} + 2\pi\sin\left(\frac{\pi t}{2}\right)\underset{\sim}{j} + \pi t\underset{\sim}{k}$ at time $t \geq 0$.

b. Find the velocity vector $\underset{\sim}{v}$ of the particle and show its speed is constant. 2 marks

c. Find the distance travelled (by calculus) and the time taken when the particle completes one cycle of its motion along the curved path. Hence show that the constant speed is the same value as that found in part b.

3 marks

d. Find the acceleration vector $\tilde{a}$ of the particle and show it is always perpendicular to its velocity vector $\tilde{v}$. Hence explain why the speed of the particle is constant.

3 marks

Consider the particle's speed changes uniformly with time, and it is doubled in its first cycle of motion.

e. Calculate the rate of change in the particle's speed using the equation found in part a or by other means.

1 mark

f. If the rate of change in the particle's speed is maintained, determine the velocity vector of the particle after completing the first two cycles.

3 marks

Question 5 (12 marks)

Six months ago a random sample of workers in a big city was taken to find the time X minutes for each travelling from home to work place.
The sample was taken over five working days (Monday to Friday). The following data were recorded.

Monday: 125 workers; mean $\overline{X} = 35$; standard deviation $s = 10$

Tuesday: 300 workers; $\overline{X} = 32$; $s = 8$

Wednesday: 230 workers; $\overline{X} = 32$; $s = 9$

Thursday: 180 workers; $\overline{X} = 36$; $s = 10$

Friday: 155 workers; $\overline{X} = 34$; $s = 9$

a. Determine the mean and standard deviation of the sample over the five working days. 3 marks

b. If the sampling of the same size was repeated at around the same time, find the probability that the mean of the repeated sample is between 33 and 34.

1 mark

c. Estimate the mean and standard deviation of travelling time X of the worker population. 1 mark

d. Determine the value of $\Pr(33 < X < 34)$. 1 mark

e. Calculate the 90% confidence interval for X. 2 marks

To investigate the effects of road works on travelling time which is expected to increase, a recent survey was conducted with sample size of 990. The mean and standard deviation of the random sample were 33.72 and 4.25 respectively.

f. A hypothesis testing is to be done. State the null hypothesis and the alternative hypothesis. 1 mark

g. Determine whether road works increase travelling time. Give evidence to support your conclusion.

2 marks

h. Describe, specifically in the context of the investigation, two types of error that are plausible. 1 mark

End of Exam 2