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Complex Numbers

Many quadratic equations $ax^2+bx+c=0$ cannot be solved within the set of real numbers *R*.

By the quadratic formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, no real solutions exist for those with discriminant $b^2-4ac<0$
$\because$ the square root of a negative number is undefined in *R*.

Example 1 $x^2+1=0$ does not have real solution(s) because $b^2-4ac=-4$ is a negative value.

Example 2 $x^2-x+1=0$ does not have real solutions because $b^2-4ac=-3$ is a negative value.

Suppose we introduce the imaginary number $\sqrt{-1}$ and label it as *i*. By definition, $i^2=-1$.

Now the two equations above can be solved in terms of *i*.

For $x^2+1=0$, $x^2=-1$, $x=\pm\sqrt{-1}$, $x=\pm i$.

For $x^2-x+1=0$, use the quadratic formula,

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{-3}}{2}=\frac{1\pm\sqrt{3}\sqrt{-1}}{2}$$
$$=\frac{1\pm i\sqrt{3}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

A number in the form $x+yi$, where $x,y\in R$ and $i^2=-1$, is called a complex number.

We use *C* to denote the set of complex numbers.
Note that $R\subset C$.

A real number can be considered as a complex number,
e.g. $5=5+0i$, $-\dfrac{2}{3}=-\dfrac{2}{3}+0i$.

We use *z* to represent a complex number, i.e. $z=x+yi$. It consists of two parts: *x* is called the real part of *z* and *y* the imaginary part of *z*. $x=\operatorname{Re}z$, $y=\operatorname{Im}z$.

Example 1 $z=2-i$, $\operatorname{Re}z=2$ and $\operatorname{Im}z=-1$.

Example 2 $z=\dfrac{1}{2}-\sqrt{3}i$, $\operatorname{Re}z=\dfrac{1}{2}$ and $\operatorname{Im}z=-\sqrt{3}$.

Equality of two complex numbers

Two complex numbers are equal if they have equal real parts <u>and</u> equal imaginary parts. The converse is also true,

i.e. $a+bi=c+di\Leftrightarrow a=c$ and $b=d$.

Example 1 Given $(x^2-3)-8i=1+(2x-x^2)i$, find *x*.

Equate the real parts, $x^2-3=1$, $x=\pm2$; equate the imaginary parts, $-8=2x-x^2$, $x^2-2x-8=0$, $x=-2$ or 4.
$x=-2$ is the only solution that satisfies both equations.

Example 2 Find *a*, *b*, given $a+b-i=3+(a-b)i$.

Equate the real parts, and the imaginary parts, $a+b=3$ and $-1=a-b$. Solve the equations simultaneously to obtain $a=1$ and $b=2$.

Addition and subtraction of complex numbers

Given $z_1=a+bi$ and $z_2=c+di$, then
$z_1\pm z_2=(a\pm c)+(b\pm d)i$.

Example 1 Given $z_1=a-bi$, $z_2=a+b+i$, $z_3=1+(a-b)i$ and $z_1+z_2-z_3=0$, find the values of a and b.

$$(a-bi)+(a+b+i)-[1+(a-b)i]=0$$
$$a+a+b-1-bi+i-(a-b)i=0$$
$$(2a+b-1)+(1-a)i=0$$
$$\therefore 2a+b-1=0 \text{ and } 1-a=0$$
$$\therefore a=1 \text{ and } b=-1.$$

Multiplication of complex numbers

Follow the usual method in algebraic expansion.

Let $z_1=a+bi$ and $z_2=c+di$, then

$$z_1z_2=(a+bi)(c+di)=ac+adi+bci+bdi^2$$
$$=(ac-bd)+(ad+bc)i.$$

Example 1 Expand $\left(\sqrt{3}-i\sqrt{2}\right)^2$.

$$\left(\sqrt{3}-i\sqrt{2}\right)^2=\left(\sqrt{3}\right)^2-2\left(\sqrt{3}\right)\left(i\sqrt{2}\right)+\left(i\sqrt{2}\right)^2$$
$$=3-i2\sqrt{6}-2=1-\left(2\sqrt{6}\right)i$$

Example 2 Simplify $\left(\sqrt{5}-2i\right)\left(\sqrt{5}+2i\right)$.

$$\left(\sqrt{5}-2i\right)\left(\sqrt{5}+2i\right)=\left(\sqrt{5}\right)^2-(2i)^2=5+4=9$$

Conjugate of a complex number

Given $z = a + bi$, then the conjugate of z, denoted as $\bar{z}$, is $\bar{z} = a - bi$. The product of z and $\bar{z}$ is a real number, (see last example 2) i.e. $z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 \in R$.

Division of complex numbers

This operation is performed by multiplying both the dividend (numerator) and the divisor (denominator) by the complex conjugate of the divisor, then simplify.

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2}$$

Example 1 Simplify $\frac{2-3i}{1+4i}$, express answer in $x + yi$ form.

$$\frac{2-3i}{1+4i} = \frac{(2-3i)(1-4i)}{(1+4i)(1-4i)} = \frac{-10-7i}{17} = -\frac{10}{17} - \frac{7}{17}i$$

Example 2 Simplify $\frac{-i}{1+\sqrt{2}-3i}$, express answer in $x + yi$ form with rational denominators.

$$\frac{-i}{1+\sqrt{2}-3i} = \frac{-i\left(1+\sqrt{2}+3i\right)}{\left(1+\sqrt{2}-3i\right)\left(1+\sqrt{2}+3i\right)} = \frac{3-\left(1+\sqrt{2}\right)i}{\left(1+\sqrt{2}\right)^2+3^2}$$

$$= \frac{3-\left(1+\sqrt{2}\right)i}{12+2\sqrt{2}} = \frac{3}{12+2\sqrt{2}} - \frac{1+\sqrt{2}}{12+2\sqrt{2}}i$$

$$= \frac{3\left(12-2\sqrt{2}\right)}{\left(12+2\sqrt{2}\right)\left(12-2\sqrt{2}\right)} - \frac{\left(1+\sqrt{2}\right)\left(12-2\sqrt{2}\right)}{\left(12+2\sqrt{2}\right)\left(12-2\sqrt{2}\right)}i$$

$$= \frac{3\left(12-2\sqrt{2}\right)}{136} - \frac{8-10\sqrt{2}}{136}i = \frac{3\left(6-\sqrt{2}\right)}{68} - \frac{4-5\sqrt{2}}{68}i$$

Multiplicative inverse of a complex number

The multiplicative inverse of z is $\frac{1}{z}$, because $z \times \frac{1}{z} = 1$. It is denoted by z^{-1}, i.e. $z^{-1} = \frac{1}{z}$.

Example 1 Find z^{-1} in $x + yi$ form, given $z = 2 - i$.

$$z^{-1} = \frac{1}{z} = \frac{1}{2-i} = \frac{1(2+i)}{(2-i)(2+i)} = \frac{2+i}{5} = \frac{2}{5} + \frac{1}{5}i$$

Powers of complex numbers

Powers of complex numbers are defined the same way as the powers of real numbers,

e.g. $z^3 = z \times z \times z$

$$z^{-2} = \left(z^{-1}\right)^2 = \left(\frac{1}{z}\right)^2 \text{ or } z^{-2} = \left(z^2\right)^{-1} = \frac{1}{z^2}$$

$$z^{\frac{1}{3}} = \sqrt[3]{z}$$

$$z^{\frac{3}{2}} = \left(\sqrt{z}\right)^3 \text{ or } \sqrt{z^3}$$

Example 1 If $z = 3 - 2i$, find $\left(\bar{z}\right)^2$.

$$\left(\bar{z}\right)^2 = (3+2i)^2 = 5 + 12i$$

Example 2 Given $z = 2 - i$, find z^{-2}.

$$z^{-2} = \left(\frac{1}{z}\right)^2 = \left(\frac{1}{2-i}\right)^2 = \left(\frac{1(2+i)}{(2-i)(2+i)}\right)^2 = \left(\frac{2+i}{5}\right)^2$$

$$= \frac{3+4i}{25} = \frac{3}{25} + \frac{4}{25}i$$

Example 3 Show that $\left(\bar{z}\right)^2 = \overline{z^2}$ and $\left(\bar{z}\right)^{-2} = \overline{z^{-2}}$, given $z = x + iy$.

$$\left(\bar{z}\right)^2 = (x-iy)^2 = \left(x^2-y^2\right) - 2xyi = \overline{\left(x^2-y^2\right)+2xyi} = \overline{z^2}.$$

$$\left(\bar{z}\right)^{-2} = \frac{1}{\left(\bar{z}\right)^2} = \frac{1}{\left(x^2-y^2\right)-2xyi}$$

$$= \frac{1\left[\left(x^2-y^2\right)+2xyi\right]}{\left[\left(x^2-y^2\right)-2xyi\right]\left[\left(x^2-y^2\right)+2xyi\right]}$$

$$= \frac{\left(x^2-y^2\right)+2xyi}{\left(x^2-y^2\right)^2+4x^2y^2}$$

$$= \frac{\left(x^2-y^2\right)+2xyi}{\left(x^2+y^2\right)^2},$$

$$\overline{z^{-2}} = \overline{\left(\frac{1}{z^2}\right)} = \overline{\left(\frac{1}{\left(x^2-y^2\right)+2xyi}\right)}$$

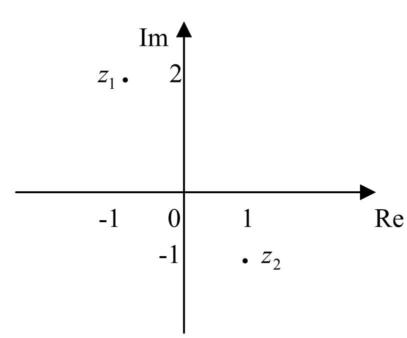
$$= \overline{\left(\frac{1\left[\left(x^2-y^2\right)-2xyi\right]}{\left[\left(x^2-y^2\right)+2xyi\right]\left[\left(x^2-y^2\right)-2xyi\right]}\right)}$$

$$= \overline{\left(\frac{\left(x^2-y^2\right)-2xyi}{\left(x^2+y^2\right)^2}\right)}$$

$$= \frac{\left(x^2-y^2\right)+2xyi}{\left(x^2+y^2\right)^2}, \quad \therefore \left(\bar{z}\right)^{-2} = \overline{z^{-2}}.$$

The complex plane (Argand plane)

A complex number can be represented as a point in the complex plane. The position of the point is indicated by the real and imaginary parts of the complex number and relative to the complex number $z = 0 + i0$, the origin of the complex plane. The horizontal axis through the origin is called the real axis and the vertical axis is the imaginary axis.



e.g. $z_1 = -1 + 2i$, $z_2 = 1 - i$.

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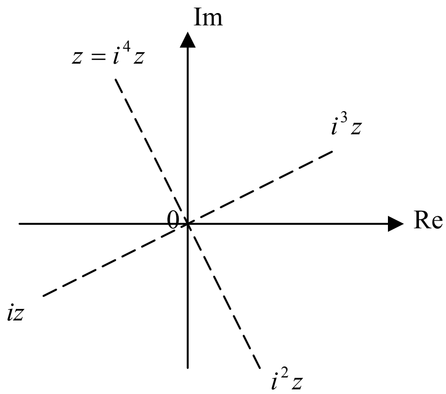
Example 1 Pick any complex number z, then find iz, i^2z, i^3z and i^4z. Plot them in the complex plane. Give a geometrical interpretation of multiplying a complex number by i.

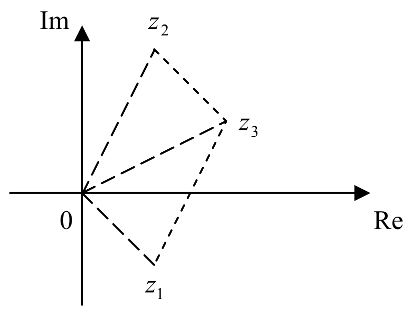
Let $z=-1+2i$, then $iz=-2-i$, $i^2z=1-2i$, $i^3z=2+i$ and $i^4z=-1+2i=z$.



Multiplying a complex number by i rotates the complex number anticlockwise by 90°. Multiplying a complex number by $-i$ rotates the complex number clockwise by 90°.

Example 2 Represent $z_1=1-i$, $z_2=1+2i$ and $z_3=z_1+z_2$ in a complex plane. What do you notice?

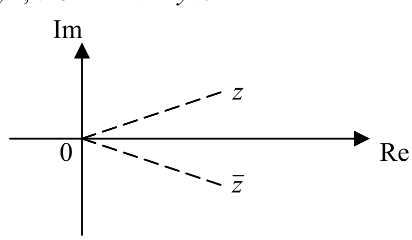
$z_3=z_1+z_2=2+i$



0, z_1, z_2 and z_3 are the vertices of a parallelogram.

Example 3 Pick any complex number z, find $\bar{z}$ and plot both in a complex plane. Discuss.

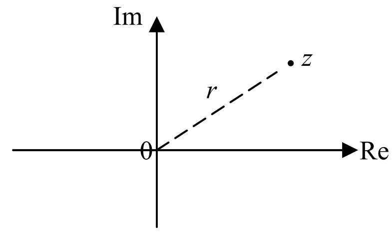
Let $z=x+yi$, then $\bar{z}=x-yi$.



$\bar{z}$ is the reflection of z in the Re-axis.

Complex numbers in polar form

The position of a complex number in the complex plane can also be described in terms of its 'distance' from the origin and the angle that the line joining the complex number to the origin makes with the positive Re-axis. It is called polar form.



The 'distance' is called the modulus of the complex number, and it is denoted by r, mod z or $|z|$.

$$r=\sqrt{(\operatorname{Re} z)^2+(\operatorname{Im} z)^2}\ .$$

The angle is called the argument of the complex number, denoted by θ or $\arg z$.

If $-\pi<\theta\le\pi$, the argument is denoted by Arg z. Arg z is called the principal value of $\arg z$.

$$\theta = \operatorname{Arg} z = \tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right),\quad \operatorname{Re} z\ne 0\ .$$

Notes: 1) Since there are two possible values for θ, it is necessary to check in which quadrant of the complex plane that z lies, and the correct value for θ is chosen accordingly.

2) If $\operatorname{Re} z=0$, then $\theta=\frac{\pi}{2}$ or $-\frac{\pi}{2}$ depending on whether $\operatorname{Im} z$ is positive or negative.

In terms of r and θ, $\operatorname{Re} z=r\cos\theta$ and $\operatorname{Im} z=r\sin\theta$, hence z in polar form is

$z=(r\cos\theta)+i(r\sin\theta)$ or $z=r(\cos\theta+i\sin\theta)$, or simply $z=rcis\theta$.

Example 1 Express $\sqrt{3}-i$ in polar form.

$r=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=\sqrt{10}$,

$\theta = \tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}$. Note: Not $\frac{5\pi}{6}$ because $\sqrt{3}-i$ is located in the fourth quadrant of the complex plane.

$\therefore \sqrt{3}-i=\sqrt{10}cis\left(-\frac{\pi}{6}\right)$.

Example 2 Express $2cis\left(-\frac{2\pi}{3}\right)$ in $x+iy$ form.

$2cis\left(-\frac{2\pi}{3}\right)=2\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right]$

$=2\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right]=-1-i\sqrt{3}$

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Example 3 Find $\mod z$ and $\operatorname{Arg} z$, given $z = -3cis\left(\frac{10\pi}{3}\right)$.

$-3cis\left(\frac{10\pi}{3}\right)$ is the rotation of $3cis\left(\frac{10\pi}{3}\right)$ anticlockwise by π, $\therefore z = -3cis\left(\frac{10\pi}{3}\right) = 3cis\left(\frac{10\pi}{3} + \pi\right)$

$$= 3cis\left(4\pi + \frac{\pi}{3}\right) = 3cis\left(\frac{\pi}{3}\right).$$

Hence $|z| = 3$ and $\operatorname{Arg} z = \frac{\pi}{3}$.

Example 4 Find $\mod \bar{z}$ and $\arg \bar{z}$, given $z = rcis\theta$.

$z = rcis\theta = r(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta)) = rcis(-\theta)$

$\therefore \mod \bar{z} = r$ and $\arg \bar{z} = -\theta$

Example 5 Find $\mod z^{-1}$ and $\arg z^{-1}$, given $z = rcis\theta$.

$z = rcis\theta = r(\cos\theta + i\sin\theta)$

$$z^{-1} = \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}cis(-\theta)$$

$\therefore \mod z^{-1} = \frac{1}{r}$ and $\arg z^{-1} = -\theta$

Multiplication and division in polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$\therefore z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

i.e. $z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$

This formula shows that to multiply two complex numbers you multiply the moduli and add the arguments.

In a similar manner a formula for division can be obtained.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}cis(\theta_1 - \theta_2), \quad z_2 \neq 0.$$

To divide two complex numbers you divide the moduli and subtract the arguments.

Example 1 Use the division formula to show that $\frac{1}{z} = \frac{1}{r}cis(-\theta)$ for $z = rcis\theta$.

$z = rcis\theta$

$$\frac{1}{z} = \frac{1cis0}{rcis\theta} = \frac{1}{r}cis(0 - \theta) = \frac{1}{r}cis(-\theta)$$

Example 2 Find the product of $1 - i$ and $\sqrt{3} + i$ in polar form.

$1 - i = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$, $\sqrt{3} + i = 2cis\left(\frac{\pi}{6}\right)$.

$$(1-i)(\sqrt{3}+i) = \left[\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right]\left[2cis\left(\frac{\pi}{6}\right)\right] = 2\sqrt{2}cis\left(-\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= 2\sqrt{2}cis\left(-\frac{\pi}{12}\right)$$

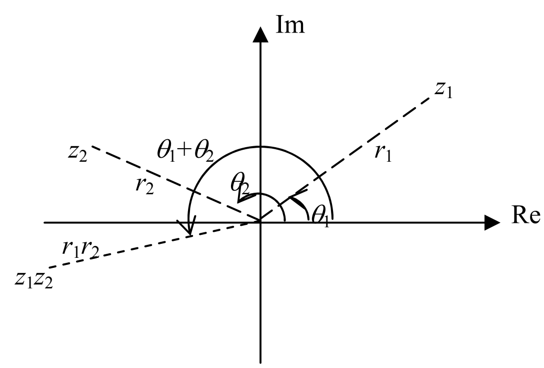
Example 3 Find the quotient $\frac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$ in polar form.

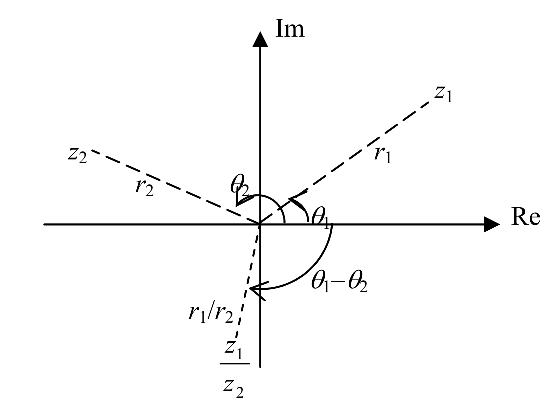
$1 - i\sqrt{3} = 2cis\left(-\frac{\pi}{3}\right)$, $1 + i\sqrt{3} = 2cis\left(\frac{\pi}{3}\right)$.

$$\frac{1 - i\sqrt{3}}{1 + i\sqrt{3}} = \frac{2cis\left(-\frac{\pi}{3}\right)}{2cis\left(\frac{\pi}{3}\right)} = 1cis\left(-\frac{\pi}{3} - \frac{\pi}{3}\right) = cis\left(-\frac{2\pi}{3}\right)$$

Geometric representation and interpretation of × and ÷ in polar form

Consider $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.





De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta) = rcis\theta$ and n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n cis(n\theta).$$

To find the nth power of a complex number you take the nth power of the modulus and multiply the argument by n.

Example 1 Use the multiplication formula for complex numbers repeatedly to show the working of De Moivre's Theorem.

Let $z = rcis\theta$,

$z^2 = zz = rrcis(\theta+\theta) = r^2 cis(2\theta)$,

$z^3 = z^2 z = r^2 rcis(2\theta+\theta) = r^3 cis(3\theta)$,

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$z^n = z^{n-1}z = r^{n-1}rcis[(n-1)\theta+\theta] = r^n cis(n\theta)$

Example 2 Find $\left(\sqrt{2}-i\sqrt{2}\right)^{10}$.

$\left(\sqrt{2}-i\sqrt{2}\right)^{10} = \left[2cis\left(-\frac{\pi}{4}\right)\right]^{10} = 2^{10}cis\left(-\frac{10\pi}{4}\right)$

$= 2^{10}cis\left(-2\pi-\frac{\pi}{2}\right) = 2^{10}cis\left(-\frac{\pi}{2}\right)$

$= 2^{10}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$

$= 2^{10}[-i] = -1024i$

***n*th roots of a complex number**

De Moivre's Theorem can also be used to find the nth roots of a complex number.

The usual notation of the nth root, $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$, for real numbers is adopted for complex numbers.

Let $w = z^{\frac{1}{n}}$, z and w in polar form are

$z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$.

$$w^n = z$$

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

Hence $s^n = r$ and $\therefore s = r^{\frac{1}{n}}$,

$$n\phi = \theta + k(2\pi) \text{ and } \therefore \phi = \frac{\theta+2k\pi}{n}.$$

Thus $\sqrt[n]{z} = z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$

$= r^{\frac{1}{n}}cis\left(\frac{\theta+2k\pi}{n}\right)$, where $k = 0,1,2,3,......$.

The nth roots of any complex number can be found by using different k values from zero up to $k = n-1$. Using $k = n$ will result in the same nth root as using $k = 0$. There are exactly n nth roots for any complex number.

Example 1 Find the cube roots of i.

There are three cube roots of i.

Change i to polar form, $i = 1cis\left(\frac{\pi}{2}\right)$.

$\sqrt[3]{i} = \sqrt[3]{1}cis\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right) = 1cis\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)$, where $k = 0,1,2$.

$k=0$, $\sqrt[3]{i} = 1cis\left(\frac{\pi}{6}\right) = 1\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right) = \frac{\sqrt{3}}{2}+\frac{1}{2}i$

$k=1$, $\sqrt[3]{i} = 1cis\left(\frac{5\pi}{6}\right) = 1\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right) = -\frac{\sqrt{3}}{2}+\frac{1}{2}i$

$k=2$, $\sqrt[3]{i} = 1cis\left(\frac{9\pi}{6}\right) = 1\left(\cos\left(\frac{9\pi}{6}\right)+i\sin\left(\frac{9\pi}{6}\right)\right) = -i$.

Example 2 Find the sixth roots of -1.

Let $-1 = 1cis(\pi)$.

$\sqrt[6]{-1} = \sqrt[6]{1}cis\left(\frac{\pi+2k\pi}{6}\right) = 1cis\left(\frac{\pi+2k\pi}{6}\right)$, where $k = 0,1,2,3,4,5$.

$k=0$, $\sqrt[6]{-1} = 1cis\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}+\frac{1}{2}i$

$k=1$, $\sqrt[6]{-1} = 1cis\left(\frac{\pi}{2}\right) = i$

$k=2$, $\sqrt[6]{-1} = 1cis\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}+\frac{1}{2}i$

$k=3$, $\sqrt[6]{-1} = 1cis\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}-\frac{1}{2}i$

$k=4$, $\sqrt[6]{-1} = 1cis\left(\frac{3\pi}{2}\right) = -i$

$k=5$, $\sqrt[6]{-1} = 1cis\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}-\frac{1}{2}i$.

Example 3 Solve $z^4+4=0$.

$z^4+4=0$, $z^4=-4$, $z=\sqrt[4]{-4}$.

Change –4 to polar form, $-4 = 4cis(\pi)$.

$z = \sqrt[4]{-4} = \sqrt[4]{4}cis\left(\frac{\pi+2k\pi}{4}\right)$, where $k = 0,1,2,3$.

$k=0$, $z = \sqrt{2}cis\left(\frac{\pi}{4}\right) = 1+i$, $k=1$, $z = \sqrt{2}cis\left(\frac{3\pi}{4}\right) = -1+i$,

$k=2$, $z = \sqrt{2}cis\left(\frac{5\pi}{4}\right) = -1-i$

$k=3$, $z = \sqrt{2}cis\left(\frac{7\pi}{4}\right) = 1-i$.

Each of the nth roots of z has the same modulus, $r^{\frac{1}{n}}$. Thus all the nth roots of z lie on the circle of radius $r^{\frac{1}{n}}$ and centre (0, 0) in the complex plane. Also, the arguments of successive roots differ by $\frac{2\pi}{n}$, $\therefore$ the nth roots of z are spaced out equally on this circle.

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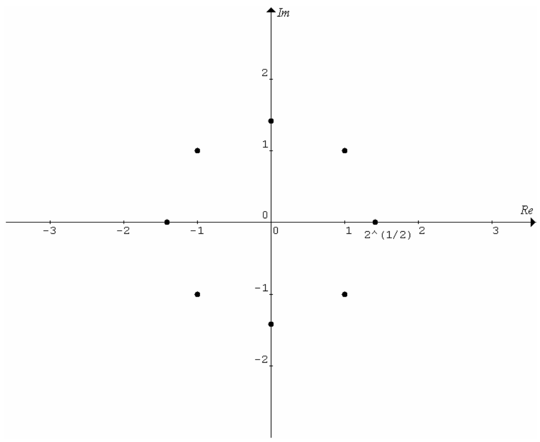
Example 4 Plot the eighth roots of 16 in the complex plane and then write down the roots in polar form and in $x+iy$ form.
One obvious eighth root of 16 is
$\sqrt[8]{16}=16^{\frac{1}{8}}=\left(2^4\right)^{\frac{1}{8}}=2^{\frac{1}{2}}=\sqrt{2}$.
The eight eighth roots of 16 have the same modulus $\sqrt{2}$ and thus lie on the circle of radius $\sqrt{2}$ and centre $(0,0)$. They are separated by $\frac{2\pi}{8}=\frac{\pi}{4}$.



Polynomials with real coefficients and factors over the set of complex number, C

Here we only consider polynomials $P(z)$ with integer coefficients only and up to degree 3, e.g.

(1) z^2+2z-3
(2) z^2+1
(3) z^2+z+1
(4) z^3-1
(5) z^3+3z^2-z-3
(6) z^3+3z^2+z+3
(7) $2z^3-4z^2+3z-1$

To factorise these polynomials, use one or a combination of the following methods: trial and error; difference of two squares; completing the square; sum or difference of two cubes; grouping; the factor theorem.

(1) $z^2+2z-3=(z+3)(z-1)$

(2) $z^2+1=z^2-i^2=(z-i)(z+i)$

(3) $z^2+z+1=z^2+z+\frac{1}{4}-\frac{1}{4}+1=\left(z+\frac{1}{2}\right)^2+\frac{3}{4}$

$=\left(z+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}i\right)^2=\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

(4) $z^3-1=z^3-1^3=(z-1)\left(z^2+z+1\right)$

$=(z-1)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

(5) $z^3+3z^2-z-3=\left(z^3+3z^2\right)-(z+3)$
$=z^2(z+3)-1(z+3)=(z+3)\left(z^2-1\right)=(z+3)(z-1)(z+1)$

(6) $z^3+3z^2+z+3=\left(z^3+3z^2\right)+(z+3)$
$=z^2(z+3)+1(z+3)=(z+3)\left(z^2+1\right)=(z+3)(z-i)(z+i)$

(7) Let $P(z)=2z^3-4z^2+3z-1$,
$P(1)=2(1)^3-4(1)^2+3(1)-1=0$, $\therefore (z-1)$ is a factor of $P(z)$.
Hence $P(z)=2z^3-4z^2+3z-1=(z-1)\left(2z^2+pz+1\right)$.
To find *p*, expand and compare coefficients, $1-p=3$, $p=-2$.
$\therefore P(z)=(z-1)\left(2z^2-2z+1\right)=2(z-1)\left(z^2-z+\frac{1}{2}\right)$

$=2(z-1)\left(z^2-z+\frac{1}{4}-\frac{1}{4}+\frac{1}{2}\right)=2(z-1)\left(\left(z-\frac{1}{2}\right)^2+\frac{1}{4}\right)$

$=2(z-1)\left(\left(z-\frac{1}{2}\right)^2-\left(\frac{1}{2}i\right)^2\right)$

$=2(z-1)\left(z-\frac{1}{2}-\frac{1}{2}i\right)\left(z-\frac{1}{2}+\frac{1}{2}i\right)$.

Linear factors of quadratic and cubic polynomials

It can be shown by completing the square that the two linear factors of a quadratic polynomial $P(z)=az^2+bz+c$ are

$z+\frac{b\pm\sqrt{b^2-4ac}}{2a}$,

i.e. $P(z)=a\left(z+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(z+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$

For quadratic polynomials with real coefficients, the two linear factors are either both over *R* or both over *C*.

For cubic polynomials with real coefficients, either all three linear factors are over *R*, or one is over R and the other two over *C*.

Also, the two linear factors over *C* always form a pair, i.e. $[z-(h+ik)][z-(h-ik)]$, where $h, k\in R$.

The above does not apply to polynomials with complex coefficients.

Example 1 Factorise $3z^2 - z + 3$ over C.

$3z^2 - z + 3$

$$= 3\left(z + \frac{-1 + \sqrt{(-1)^2 - 4(3)(3)}}{2(3)}\right)\left(z + \frac{-1 - \sqrt{(-1)^2 - 4(3)(3)}}{2(3)}\right)$$

$$= 3\left(z + \frac{-1 + \sqrt{-35}}{6}\right)\left(z + \frac{-1 - \sqrt{-35}}{6}\right)$$

$$= 3\left(z + \frac{-1 + i\sqrt{35}}{6}\right)\left(z + \frac{-1 - i\sqrt{35}}{6}\right)$$

Example 2 Given that $z + 1 - 2i$ is a factor of $z^3 + pz^2 + qz - 5$, find the values of p and q, given $p, q \in R$.

Since the polynomial has real coefficients and $z + 1 - 2i$ is a factor, $\therefore z + 1 + 2i$ is also a factor, and the third factor is real.

Hence $z^3 + pz^2 + qz - 5 = (z + 1 - 2i)(z + 1 + 2i)(z + r)$, $r \in R$

Expand, collect like terms and compare coefficients,

$z^3 + pz^2 + qz - 5 = z^3 + (r + 2)z^2 + (2r + 5)z + 5r$,

$\therefore r = -1$, $p = r + 2 = 1$, $q = 2r + 5 = 3$.

Example 3 Show that $z - 1 + i$ is a factor of $z^3 + 2z^2 - 6z + 8$. Find the other linear factors of the cubic polynomial.

Let $P(z) = z^3 + 2z^2 - 6z + 8$.

$P(1 - i) = (1 - i)^3 + 2(1 - i)^2 - 6(1 - i) + 8$

$= 1 - 3i - 3 + i - 4i - 6 + 6i + 8 = 0$, $\therefore (z - 1 + i)$ is a factor.

The polynomial has real coefficients, $\therefore (z - 1 - i)$ is also a factor.

Hence $P(z) = (z - 1 + i)(z - 1 - i)(z - r)$, where $r \in R$.

Expand the pair of conjugates and compare,

$z^3 + 2z^2 - 6z + 8 = (z^2 - 2z + 2)(z - r)$, $r = -4$.

The third factor is $(z + 4)$.

Factorisation of polynomials of the form $z^4 - a$, $z^4 + a$, $z^6 - a$

Example 1 Factorise $2z^4 - 32$ over C.

$2z^4 - 32 = 2(z^4 - 16) = 2(z^2 - 4)(z^2 + 4)$

$= 2(z - 2)(z + 2)(z - 2i)(z + 2i)$

Example 2 Factorise $z^4 + 4$ over C.

Complete the square by adding and subtracting $4z^2$,

$z^4 + 4 = z^4 + 4z^2 + 4 - 4z^2$

$= (z^2 + 2)^2 - (2z)^2$

$= (z^2 - 2z + 2)(z^2 + 2z + 2)$

$= (z - 1 - i)(z - 1 + i)(z + 1 - i)(z + 1 + i)$

Example 3 Factorise $z^6 - 64$ over C.

$z^6 - 64 = (z^3 - 8)(z^3 + 8)$

$= (z - 2)(z^2 + 2z + 4)(z + 2)(z^2 - 2z + 4)$

$= (z - 2)(z + 1 - i\sqrt{3})(z + 1 + i\sqrt{3})(z + 2)(z - 1 - i\sqrt{3})(z - 1 + i\sqrt{3})$

Polynomial equations with real coefficients

Let $P_n(z) = 0$ be a n-degree polynomial equation.

Quadratic equations $P_2(z) = 0$ can be solved with the quadratic formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Higher degree polynomial equations can be solved by changing the polynomials to factorised form over C.

For polynomial equations with real coefficients, the non-real roots always occur in conjugate pairs. This is known as the **conjugate root theorem**. Note: The conjugate root theorem is applicable to polynomial equations with real coefficients only.

The **fundamental theorem of algebra** states that if $n > 0$ and the coefficients are either real or complex numbers, there is at least one complex number α_1 such that $P_n(\alpha_1) = 0$. α_1 is called a root of $P_n(z)$. According to the **factor theorem**, $z - \alpha_1$ is a factor of $P_n(z)$ and hence

$$P_n(z) = (z - \alpha_1)Q_{n-1}(z).$$

Apply the fundamental theorem of algebra and the factor theorem to $Q_{n-1}(z)$, then

$$P_n(z) = (z - \alpha_1)(z - \alpha_2)Q_{n-2}(z).$$

Repeated application of the two theorems enables one to show that $P_n(z)$ has n linear factors over C,

$$P_n(z) = a(z - \alpha_1)(z - \alpha_2)\ldots\ldots(z - \alpha_n),$$

$\therefore P_n(z) = 0$ has n solutions. They are the roots of $P_n(z)$. It is possible that some of the solutions (roots) may be repeated (i.e. the same).

Example 1 Solve $z^3 - z^2 + z - 1 = 0$ over C.

$z^3 - z^2 + z - 1 = 0$, $(z^3 - z^2) + (z - 1) = 0$,

$z^2(z - 1) + 1(z - 1) = 0$, $(z - 1)(z^2 + 1) = 0$,

$(z - 1)(z - i)(z + i) = 0$, $\therefore z = 1, i, -i$.

$z^3 - z^2 + z - 1 = 0$ is cubic, $\therefore$ it has 3 solutions. Since its coefficients are real, $\therefore$ its two complex solutions are conjugates.

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Example 2 Use the fundamental theorem of algebra to solve $z^3-(2-i)z^2+z-2+i=0$.

Let $P(z)=z^3-(2-i)z^2+z-2+i$.

$P(i)=i^3-(2-i)i^2+i-2+i=-i+2-i+i-2+i=0$, $\therefore i$ is a root of $P(z)=z^3-(2-i)z^2+z-2+i$.

$\therefore P(z)=z^3-(2-i)z^2+z-2+i=(z-i)(z^2+pz+q)$

$=z^3+(p-i)z^2+(q-ip)z-iq$.

Compare coefficients, $p-i=-2+i$ and $-iq=-2+i$.

$\therefore p=-2+2i$ and $q=-1-2i$.

Hence

$P(z)=(z-i)(z^2+pz+q)=(z-i)(z^2+(-2+2i)z+(-1-2i))$

Use the quadratic formula to find the other two roots:

$$z=\frac{-(-2+2i)\pm\sqrt{(-2+2i)^2-4(1)(-1-2i)}}{2(1)}=\frac{2-2i\pm 2}{2}$$

$=2-i$ or $-i$.

The three roots are $i,-i,2-i$.

Alternative method: Factorisation by grouping.

$z^3-(2-i)z^2+z-2+i=0$

$(z^3-(2-i)z^2)+(z-2+i)=0$

$z^2(z-2+i)+1(z-2+i)=0$

$(z^2+1)(z-2+i)=0$, $(z-i)(z+i)(z-2+i)=0$,

$\therefore z=i,-i,2-i$.

Example 3 Show that $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ is a root of the equation $z^2-i=0$. Find the other root(s).

Let $P(z)=z^2-i$,

$$P\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^2-i=\left(\frac{1}{2}+i-\frac{1}{2}\right)-i=0,$$

$\therefore \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ is a root of $z^2-i=0$ and $z-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$ is a factor of z^2-i.

According to the fundamental theorem of algebra, $z^2-i=0$ is second degree, $\therefore$ it has exactly two roots. Let $a+bi$ be the other root.

$$\therefore z^2-i=\left[z-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\right][z-(a+bi)]$$

$$=z^2-\left[\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)+(a+bi)\right]z-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)(a+bi)$$

Compare the coefficient of z on both sides,

$\therefore \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)+(a+bi)=0$, $\therefore a=-\frac{1}{\sqrt{2}}$ and $b=-\frac{1}{\sqrt{2}}$.

The second root is $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$.

Alternative methods in finding the roots of $z^2-i=0$:

(a) Using polar form.

$z^2-i=0$, $\therefore z^2=i$, $z=\sqrt{i}=\sqrt{1cis\left(\frac{\pi}{2}+2k\pi\right)}$,

$k=0$, $z=\sqrt{i}=\sqrt{1}cis\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$.

$k=1$, $z=\sqrt{i}=\sqrt{1}cis\left(\frac{5\pi}{4}\right)=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$.

(b) Let $z=a+bi$ be the root of $z^2-i=0$.

$\therefore (a+bi)^2-i=0$.

Expand, collect real and imaginary parts to obtain $(a^2-b^2)+(2ab-1)i=0$.

$\therefore a^2-b^2=0$ and $2ab-1=0$.

Solve simultaneously to obtain

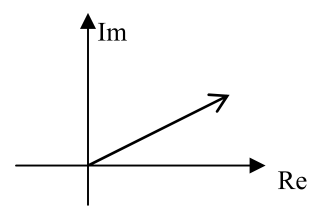
$a=\frac{1}{\sqrt{2}}$ and $b=\frac{1}{\sqrt{2}}$, or $a=-\frac{1}{\sqrt{2}}$ and $b=-\frac{1}{\sqrt{2}}$.

Note: The last example shows that for quadratic equation of the form $z^2-c=0$, $c\in C$, if α is a root, then the other root is $-\alpha$.

Representation of relations and regions in the complex plane

A ray in the complex plane: It is a set of complex numbers having the same argument,

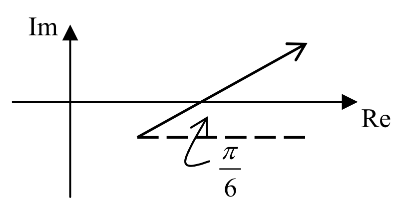
e.g. $\left\{z: Argz=\frac{\pi}{6}\right\}$



Note: $0+0i=0cis\left(\frac{\pi}{6}\right)$, $\therefore 0+0i\in\left\{z: Argz=\frac{\pi}{6}\right\}$.

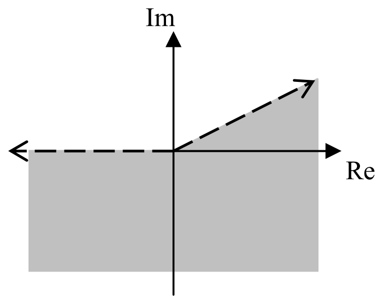
This ray can be translated horizontally and/or vertically,

e.g. $\left\{z: Arg(z-(2-i))=\frac{\pi}{6}\right\}$, in this case the starting point of the ray is $2-i$ instead of $0+0i$.

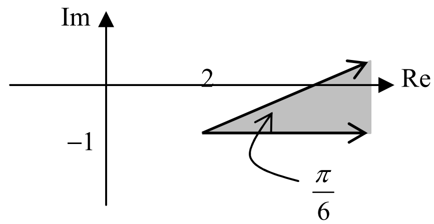


Region defined by a ray

Example 1 Sketch the region defined by $\left\{z : Argz < \dfrac{\pi}{6}\right\}$.

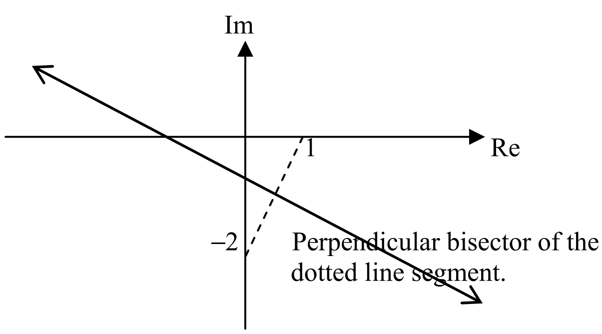


Example 2 Sketch the region defined by
$\left\{z : 0 \le Arg\left(z-(2-i)\right) \le \dfrac{\pi}{6}\right\}$.



A line in the complex plane: One way to define a straight line in the complex plane is to consider it as a set of complex numbers such that each number is 'equidistant' from two fixed numbers, e.g. $\{z : |z-1| = |z+2i|\}$

$|z-1|$ is the 'distance' of z from $1+0i$, and $|z+2i|$ is the 'distance' of z from $0-2i$.



Let $z = x+iy$. $|z-1| = |z+2i|$ becomes

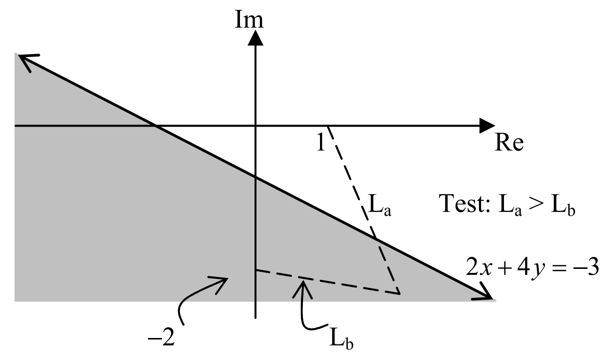
$$|x+iy-1| = |x+iy+2i|$$
$$|(x-1)+iy| = |x+i(y+2)|$$
$$(x-1)^2 + y^2 = x^2 + (y+2)^2$$
$$2x+4y = -3$$

The last equation is known as the cartesian equation of the line defined by $\{z : |z-1| = |z+2i|\}$.

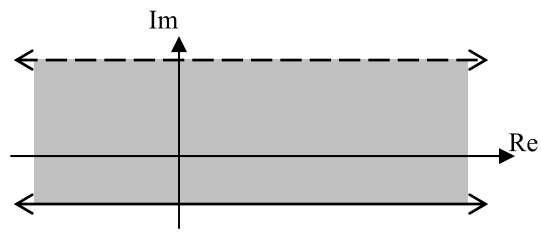
Since $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ one can also define the line as $\{z : 2\operatorname{Re} z + 4\operatorname{Im} z = -3\}$.

Region defined by a line

Example 1 Sketch the region defined by $\{z : |z-1| \ge |z+2i|\}$.

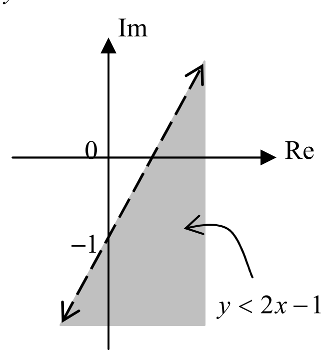


Example 2 Sketch $\{z : -1 \le \operatorname{Im} z < 2\}$.

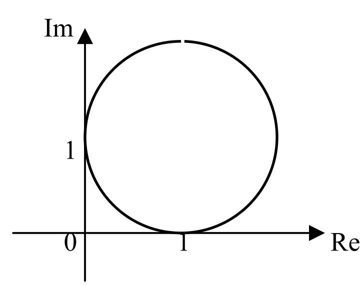


Example 3 Sketch $\{z : 2\operatorname{Re} z - \operatorname{Im} z > 1\}$.

Easier to find the cartesian inequation first before sketching.
$2x - y > 1$, $\therefore\ y < 2x - 1$.



A circle in the complex plane: It can be defined as a set of complex numbers that are at the same 'distance' from a particular complex number (the centre),
e.g. $\{z : |z-(1+i)| = 1\}$, radius = 1, centre is (1, 1).



Example 1 Show that $\{z : 2|z+1| = |z-i|\}$ also defines a circle in the complex plane.

Let $z = x + iy$, $2|(x+1)+iy| = |x+(y-1)i|$,

$4|(x+1)+iy|^2 = |x+(y-1)i|^2$, $4\left((x+1)^2 + y^2\right) = x^2 + (y-1)^2$,

$4x^2 + 8x + 4 + 4y^2 = x^2 + y^2 - 2y + 1$,

$3x^2 + 8x + 3y^2 + 2y = -3$,

$x^2 + \frac{8}{3}x + y^2 + \frac{2}{3}y = -1$,

$x^2 + \frac{8}{3}x + \left(\frac{4}{3}\right)^2 + y^2 + \frac{2}{3}y + \left(\frac{1}{3}\right)^2 = -1 + \left(\frac{4}{3}\right)^2 + \left(\frac{1}{3}\right)^2$

$\left(x+\frac{4}{3}\right)^2 + \left(y+\frac{1}{3}\right)^2 = \frac{8}{9}$. It is a circle of radius $\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$ centred at $\left(-\frac{4}{3}, -\frac{1}{3}\right)$.

Example 2 Sketch the region $\{z : 2|z+1| < |z-i|\}$.

Let $z = x + iy$, $2|(x+1)+iy| < |x+(y-1)i|$,

$4|(x+1)+iy|^2 < |x+(y-1)i|^2$, $4\left((x+1)^2 + y^2\right) < x^2 + (y-1)^2$,

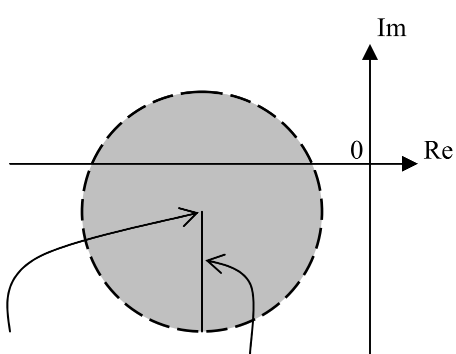
$4x^2 + 8x + 4 + 4y^2 < x^2 + y^2 - 2y + 1$,

$3x^2 + 8x + 3y^2 + 2y < -3$,

$x^2 + \frac{8}{3}x + y^2 + \frac{2}{3}y < -1$,

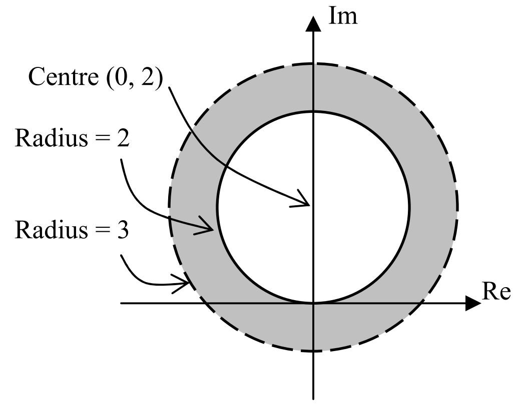
$x^2 + \frac{8}{3}x + \left(\frac{4}{3}\right)^2 + y^2 + \frac{2}{3}y + \left(\frac{1}{3}\right)^2 < -1 + \left(\frac{4}{3}\right)^2 + \left(\frac{1}{3}\right)^2$

$\left(x+\frac{4}{3}\right)^2 + \left(y+\frac{1}{3}\right)^2 < \frac{8}{9}$.

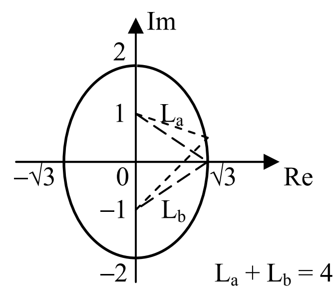


Centre $\left(-\frac{4}{3}, -\frac{1}{3}\right)$ Radius $= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

Example 3 Sketch the region $\{z : 2 \le |z-2i| < 3\}$.



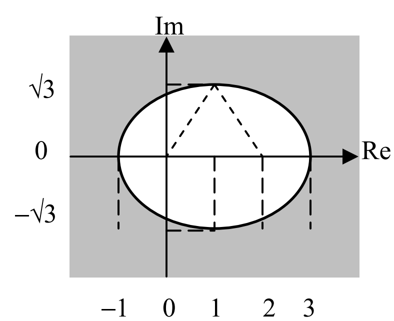
An ellipse in the complex plane: It can be defined as a set of complex numbers such that the sum of 'distances' from two given complex numbers is constant and greater than the 'distance' between the two given complex numbers, e.g. $\{z : |z+i| + |z-i| = 4\}$. In this case, the two given complex numbers are $-i$ and i.



Example 1 Sketch the region $\{z : |z| + |z-2| \ge 4\}$.

Let L_a be the 'distance' of z from 0, and L_b the 'distance' of z from 2. $L_a + L_b \ge 4$

$|z| + |z-2| = 4$ is the ellipse and $|z| + |z-2| > 4$ is the region outside the ellipse.



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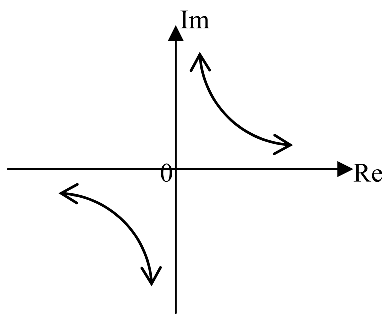
Other simple curves:

Example 1 Sketch the Argand diagram of $\{z : \operatorname{Re} z \times \operatorname{Im} z = 1\}$.

Easier to find the cartesian equation first before sketching.

Let $z = x + iy$, $\operatorname{Re} z \times \operatorname{Im} z = 1$, $xy = 1$, $y = \dfrac{1}{x}$.



Example 2 Sketch the Argand diagram of $\{z : |\operatorname{Im}(z + 2i)| = |z - 2i|\}$.

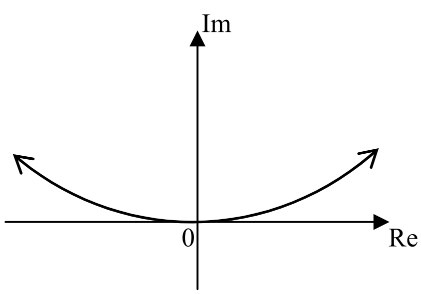
Find the cartesian equation first before sketching.

Let $z = x + iy$, $|\operatorname{Im}(z + 2i)| = |z - 2i|$,

$|\operatorname{Im}(x + (y + 2)i)| = |x + (y - 2)i|$, $\therefore |y + 2| = |x + (y - 2)i|$,

$|y + 2|^2 = |x + (y - 2)i|^2$, i.e. $(y + 2)^2 = x^2 + (y - 2)^2$.

Expand and simplify, $y = \dfrac{1}{8}x^2$.



Example 3 Sketch the Argand diagrams of $\{z : |\operatorname{Im}(z + 2i)| = 2|z - 2i|\}$.

Let $z = x + iy$, $|\operatorname{Im}(z + 2i)| = 2|z - 2i|$,

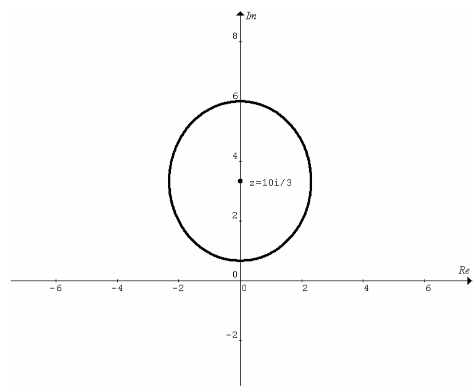
$|\operatorname{Im}(x + (y + 2)i)| = 2|x + (y - 2)i|$, $\therefore |y + 2| = 2|x + (y - 2)i|$,

$|y + 2|^2 = 4|x + (y - 2)i|^2$, i.e. $(y + 2)^2 = 4\left(x^2 + (y - 2)^2\right)$.

Expand, simplify and complete the square,

$$4x^2 + 3\left(y - \frac{10}{3}\right)^2 = \frac{64}{3}, \text{ or } \frac{x^2}{\left(\frac{4\sqrt{3}}{3}\right)^2} + \frac{\left(y - \frac{10}{3}\right)^2}{\left(\frac{8}{3}\right)^2} = 1.$$

It is an ellipse centred at $\left(0, \dfrac{10}{3}\right)$, i.e. $z = \dfrac{10}{3}i$.



Example 4 Sketch the Argand diagrams of $\{z : 2|\operatorname{Im}(z + 2i)| = |z - 2i|\}$.

Let $z = x + iy$, $2|\operatorname{Im}(z + 2i)| = |z - 2i|$,

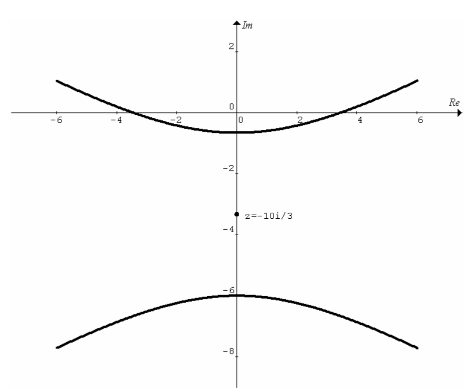
$2|\operatorname{Im}(x + (y + 2)i)| = |x + (y - 2)i|$, $\therefore 2|y + 2| = |x + (y - 2)i|$,

$4|y + 2|^2 = |x + (y - 2)i|^2$, i.e. $4(y + 2)^2 = x^2 + (y - 2)^2$.

Expand, simplify and complete the square,

$$x^2 - 3\left(y + \frac{10}{3}\right)^2 = -\frac{64}{3}, \text{ or } \frac{x^2}{\left(\frac{8\sqrt{3}}{3}\right)^2} - \frac{\left(y + \frac{10}{3}\right)^2}{\left(\frac{8}{3}\right)^2} = -1.$$

It is a hyperbola centred at $\left(0, -\dfrac{10}{3}\right)$, i.e. $z = -\dfrac{10}{3}i$.



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Example 5 Sketch the Argand diagram of $\{z : |z+5| - |z-5| = 8\}$.

Let $z = x + iy$, $|z+5| - |z-5| = 8$, $|z+5| = |z-5| + 8$,

$|(x+5)+iy|^2 = (|(x-5)+iy| + 8)^2$,

$|(x+5)+iy|^2 = |(x-5)+iy|^2 + 16|(x-5)+iy| + 64$,

$(x+5)^2 + y^2 = (x-5)^2 + y^2 + 16|(x-5)+iy| + 64$.

Expand and simplify, $5x - 16 = 4|(x-5)+iy|$.

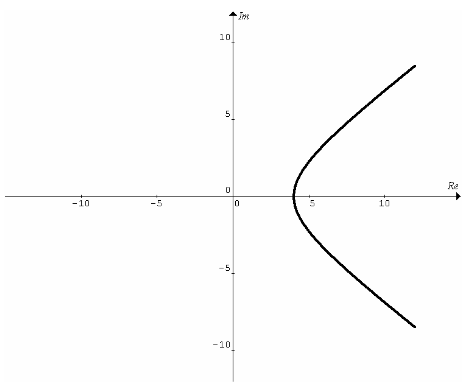
$(5x-16)^2 = (4|(x-5)+iy|)^2$,

$25x^2 - 160x + 256 = 16((x-5)^2 + y^2)$,

$25x^2 - 160x + 256 = 16x^2 - 160x + 400 + 16y^2$,

$\therefore 9x^2 - 16y^2 = 144$ or $\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$.

It is a hyperbola centred at the origin. $\{z : |z+5| - |z-5| = 8\}$ is the right hand branch only! Why?

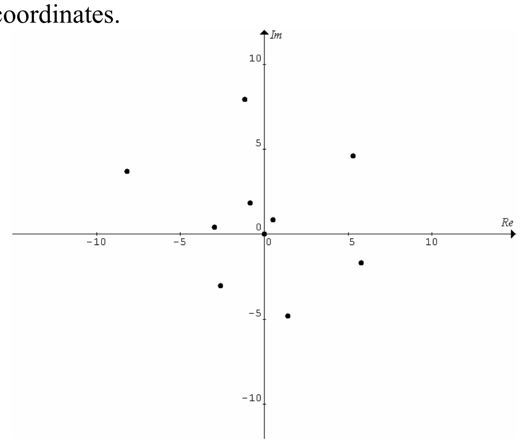


Example 6 Plot $\{z : |z| = \arg z\}$ in the complex plane.

Set up a table of values for $r = |z|$ and $\theta = \arg z$ in radians.

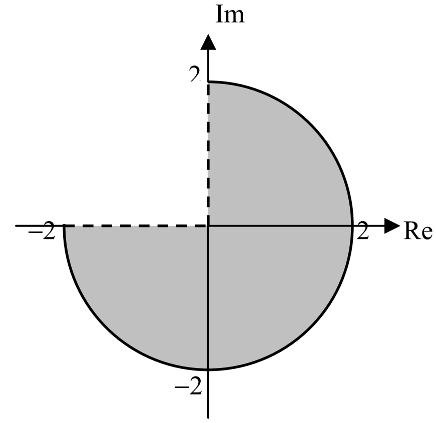
r	0	1	2	3	4	5	6	7	8	9
θ	0	1	2	3	4	5	6	7	8	9

Polar coordinates.



$\{z : |z| = \arg z\}$ is a spiral.

Example 7 Sketch in the complex plane the region defined by $\{z : |z| \le 2\} \cap \left\{z : Argz < \frac{\pi}{2}\right\}$.



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